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On an Application of Lambert's *W* Function to Infinite Exponentials

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In this article we survey the basic results concerning the convergence of Infinite Exponentials; we use Lambert's W function to show convergence for the real and complex cases in a more elegant way and prove several incidental results about Infinite Exponentials. We also show how to extend analytically the Infinite Exponential function over the complex plane and how to derive exact expansions for finite and infinite power iterates of the hyperpower function. As a final application we derive several series identities involving Infinite Exponentials.

Keywords: Lambert's W function; Infinite exponential; Power tower; Iteration; Entire function; Fixed point; Periodic point; Fatou set; Julia set; Analytic continuation; Auxiliary equation

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INTRODUCTION

Lambert's W or ω function has acquired popularity only recently, due to advances in computational mathematics. Although compositions of this function appear in a disguised form in Barrow [7, p. 153], De Villiers and Robinson [42, p. 14] and Knoebel [24, p. 235], most of W's essential properties are presented in Corless *et al.* [16, pp. 344–349] and [17, pp. 2–8]. Some of these properties can be used to greatly simplify the answer to the problem of when infinite exponentials converge.

1. NOTATION

We work with the principal branch of the complex log function, and use Maurer's notation for successive power iterates and the infinite iterate (see Knoebel [24, pp. 239–240]).

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Definition 1.1 For $z \in \mathbb{C} \setminus \{x \in \mathbb{R}: x \leq 0\}$ and $n \in \mathbb{N}$,

$$ {}^{n}z = \begin{cases} z, & \text{if } n = 1, \\ z^{({}^{n-1}z)}, & \text{if } n > 1. \end{cases} $$

Definition 1.2 Whenever the following limit exists and is finite,

$$ {}^{\infty}z = \lim_{n \to \infty} {}^{n}z. $$

We also use the exponential function $g(z)$ and its iterates:

$$ g(z) = c^{z} \tag{1.1} $$

$$ g^{(n)}(z) = \begin{cases} g(z) & \text{if } n = 1, \\ g\big(g^{(n-1)}(z)\big) & \text{if } n > 1. \end{cases} \tag{1.2} $$

${}^{n}z$ and $g^{(n)}(z)$ are related: ${}^{n}c = g^{(n)}(1)$. We will also use the following function:

Definition 1.3 $m(z) = z\mathrm{e}^{z},\ z \in \mathbb{C}$.

We specify the coefficients of the series for successive power iterates of the exp function, as $a_{m,n}$.

Definition 1.4

$$ {}^{m}(\mathrm{e}^{z}) = \sum_{n=0}^{\infty} a_{m,n} z^{n}. $$

Equations with complex exponents throughout this article are always understood to use the principal branch of complex exponentiation, whenever necessary: $c^{w} = \mathrm{e}^{w \log(c)}$, $c \neq 0$, with log being the principal branch of the complex log function.

The real counterparts of all functions in this section will be denoted as having real arguments x instead of z to avoid any confusion.

Whenever a complex function is multi-valued and a parameter $k \in \mathbb{Z}$ is required to indicate the branch chosen, the omission of k altogether indicates always the principal branch of this function $(k = 0)$.

The complex unit disk will be denoted by D unless specified otherwise.

For the term *analytic function* we use the definition found in Churchill and Brown in [15, p. 46].

The term *Infinite Exponential* was coined by Barrow in [7, p. 150]. Briefly, it is the infinite tower $z_1^{z_2^{z_3^{\cdot^{\cdot^{\cdot}}}}}$, with $z_n \in \mathbb{R}$ (or $z_n \in \mathbb{C}$), $\forall n \in \mathbb{N}$. In this paper we concern ourselves with the case where $z_n = z$, $\forall n \in \mathbb{N}$. Accordingly $z^{z^{\cdot^{\cdot^{\cdot^{z}}}}}$ (whether real or complex) will be used interchangeably with $z^{\left(z^{\cdot^{\cdot^{\cdot^{z}}}}\right)}$ to denote repeated exponentiation from top to bottom. In most cases we will denote finite exponentials using Definition 1.1 and infinite exponentials using Definition 1.2.

De Villiers and Robinson in [42, p. 13] introduce the term *nth auxiliary equation* for the equations $g^{(n)}(z) = z$. In particular, $c^{z} = z$ is referred to as the *first auxiliary equation* and $c^{c^z} = z$ is referred to as the *second auxiliary equation*, when dealing with infinite exponentials.

For the terminology about *periodic points* of functions, *normality*, *Fatou* and *Julia* sets, we refer the reader to Bergweiler [9, pp. 2–3] and Branner [11, pp. 37–41].

For an introduction to *fractals*, *dynamical systems* in general, *Mandelbrot* and *Julia* sets we refer the reader to Peitger and Richter [34, pp. 27–52], Barnsley [6, pp. 248, 303], and Branner [11, pp. 37–41]. For specific iterative processes and computer implementations that generate Mandelbrot and Julia sets, Barnsley [6, pp. 251–320] and Becker and Dörfler [8, pp. 92–145]. For successively more formal definitions, theory, and terminology behind dynamical systems, Peitgen and Richter [34, pp. 27–46], Barnsley [6, pp. 302–320], Bergweiler [10, pp. 2–26] and Milnor [30, Sections 3.1–3.5], all of which contain extensive bibliographies.

2. LAMBERT'S *W* FUNCTION

Definition 2.1 The complex function W is the function which solves for z the equation:

$$m(z) = w,\ w,\ z \in \mathbb{C}$$

or alternatively:

Definition 2.2 The complex function W satisfies the functional equation:

$$W(z)\mathrm{e}^{W(z)} = z, z \in \mathbb{C}.$$

W is multi-valued and as such it has many branches. It is usually denoted as $W(k, z)$, with $k \in \mathbb{Z}$ specifying the appropriate complex branch chosen. In particular, the principal branch of this function corresponding to $k = 0$, $W(0, z)$ will be denoted as $W(z)$ and the corresponding real valued function will be denoted as $W(x)$.

Some useful properties of W follow. Most of these can be found in [17] and can be validated numerically with Maple in Redfern [35, p. 305]. We selectively prove those which are not explicitly proved in [17] or [16].

For $k \in \mathbb{Z}$, the various branches of $W(k, z)$ are defined using the branch cuts BC_k and counterclockwise continuity (CCC) around the corresponding branch points:

$$BC_k = \begin{cases} \left(-\infty,\ -\mathrm{e}^{-1}\right), & \text{if } k = 0 \\ \left(-\infty,\ -\mathrm{e}^{-1}\right) \cup \left(-\mathrm{e}^{-1}, 0\right), & \text{if } k = -1 \\ (-\infty, 0), & \text{otherwise} \end{cases}.$$

Note that the branch point $z_0 = -\mathrm{e}^{-1}$ of $W(z)$ is $m(z)$, where z satisfies: $dm/dz = 0$. This branch point is shared between $W(z)$ and $W(-1, z)$.

Let $CN = (-\infty, -1)$ and consider the set of curves:

$$C_k = \begin{cases} -y\cot(y) + yi, y \in (2k\pi, (2k+1)\pi), & \text{if } k \geq 0, \\ -y\cot(y) + yi, y \in ((2k+1)\pi, (2k+2)\pi), & \text{if } k < 0. \end{cases}$$

LEMMA 2.3 *The image of the branch cut BC_k of $W(k,z)$ under $W(k,z)$ is always*:

$$W(k, BC_k) = \begin{cases} C_{-1} \cup CN, & \text{if } k = -1 \\ C_k, & \text{otherwise} \end{cases}$$

Now define the domains D_k as follows:

$$D_k = \begin{cases} \text{region between } C_1, CN, C_0, & \text{if } k = 1 \\ \text{region between } C_{-1}, CN, C_{-2}, & \text{if } k = -1 \\ \text{region between } C_k, C_{k-1}, & \text{otherwise} \end{cases}$$

The curves C_k and domains D_k are shown in Fig. 1.

LEMMA 2.4 $W(k,z) \in D_k,\ k \in \mathbb{Z},\ z \in \mathbb{C}$.

COROLLARY 2.5 $W(z) \in D_0$, $z \in \mathbb{C}$.

LEMMA 2.6 $W(k, m(z)) = z$, $k \in \mathbb{Z}$, $z \in D_k$ *and* $m(W(k,z)) = z$, $k \in \mathbb{Z}$, $z \in \mathbb{C}$.

COROLLARY 2.7 $W(m(z)) = z$, $z \in D_0$ *and* $m(W(z)) = z$, $z \in \mathbb{C}$.

From the symmetry of the C_ks and D_ks follow:

LEMMA 2.8 $\overline{W(k,z)} = W(-k, \bar{z})$, $k \in \mathbb{Z}$, $z \in \mathbb{C}$.

COROLLARY 2.9 $\overline{W(z)} = W(\bar{z})$, $z \in \mathbb{C}$.

Since $(-\infty, -1] \in D_{-1}$ and $[-1, +\infty) \in D_0$, only the branches corresponding to $k=0$ and $k=-1$ can ever assume real values:

LEMMA 2.10 $W(k,z) \in \mathbb{R} \Rightarrow k \in \{-1, 0\}$.

LEMMA 2.11 *$W(x)$ is real valued, continuous and strictly increasing on the interval* $[-e^{-1}, +\infty)$.

LEMMA 2.12 *$W(-1,x)$ is real valued, continuous and strictly decreasing on the interval* $[-e^{-1}, 0)$.

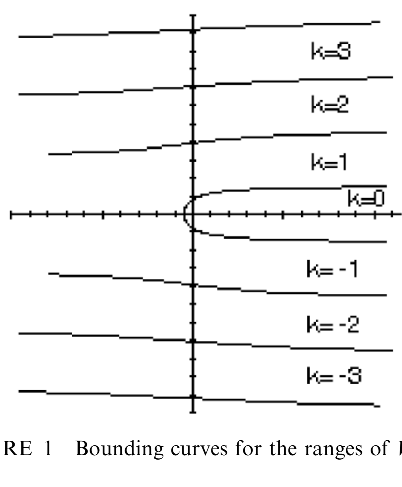

FIGURE 1 Bounding curves for the ranges of $W(k,z)$.

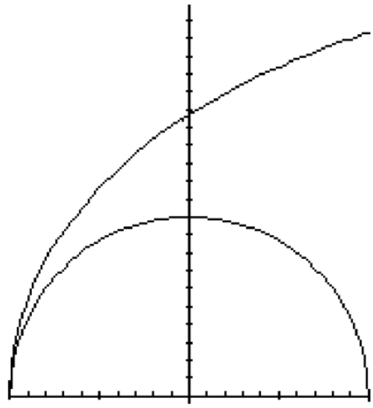

FIGURE 2 $y\cot(y) + yi, y \in (0, \pi)$ and half unit circle.

Lemmas 2.13 and 2.14 follow by considering $m(z)$ and the fact that $-e^{-1}$ is a shared branch point between $W(z)$ and $W(-1, z)$:

LEMMA 2.13 $W(e) = 1$.

LEMMA 2.14 $W(-e^{-1}) = W(-1, -e^{-1}) = -1$.

LEMMA 2.15 *$D \subset D_0$ and $\partial D \cap \partial D_0 = \{-1\}$.*

Proof It suffices to show $\left|-y\cot(y) + yi\right| > 1$ for all $y \in (0, \pi/2)$, which is easily shown using elementary calculus (Fig. 2). $\lim_{y \to 0^+}(-y\cot(y) + yi) = -1 \in \partial D \cap \partial D_0$ while $z \in \partial D \cap \partial D_0 \Rightarrow \cos(\pi - y) + \sin(\pi - y)i = -y\cot(y) + yi \Rightarrow \{\sin(y) = y, -y\cot(y) = \cos(\pi - y)\}$. From the first equation we get $y = k\pi$, $k \in \mathbb{Z}$. From those only the $y = (2k+1)\pi$, $k \in \mathbb{Z}$ satisfy the second equation as a limit, so $z = \cos((2k+1)\pi) + \sin((2k+1)\pi)i = -1$ and the lemma follows. ∎

LEMMA 2.16 *$W(z)$ is analytic at $z_0 = 0$ with series expansion:*

$$S(z) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1} z^n}{n!}$$

and radius of convergence: $R_s = e^{-1}$.

Proof Details about the expansion $S(z)$ as well as various other expansions are given in [16]. The Ratio Test reveals the radius of convergence.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\left(1 + \frac{1}{n}\right)^{n-1} z\right| = |ez| < 1,$$

or equivalently,

$$|z| < e^{-1}.$$

$S(z)$ is actually valid on the entire disk $D_w = \{z : |z| \leq \mathrm{e}^{-1}\}$. If $|z| = \mathrm{e}^{-1}$, then

$$\left|\frac{(-n)^{n-1}z^n}{n!}\right| = \frac{n^{n-1}}{\mathrm{e}^n n!} < \frac{n^{n-1}}{\sqrt{2\pi}n^{(n+1/2)}} = \frac{1}{\sqrt{2\pi}n^{3/2}},$$

using Stirling's approximation, and the series $\sum_{n=1}^{\infty} 1/\sqrt{2\pi}n^{3/2}$ is convergent. ■

It also follows that the radius of convergence of $S(z)$ is $R_s = \mathrm{e}^{-1}$, since $W(z)$ has a branch point at $z_0 = -\mathrm{e}^{-1}$.

3. THE CENTRAL LEMMA

Lemma 3.3 is mentioned (without proof) in [16, p. 332], but the author feels it needs some careful justification, particularly in view of the multi-valued nature of W.

$g(z)$ in (1.1) is intimately tied with infinite exponentials. In general, given $c \in \mathbb{C}$, $c \notin \{0, 1\}$, if the sequence $\{g^{(n)}(z)\}_{n \in \mathbb{N}}$ converges, it must converge to a fixed point of $g(z)$ or equivalently the limit must satisfy the first auxiliary equation,

$$z = g(z). \tag{3.1}$$

Equation (3.1) can always be solved analytically via W.

Lemma 3.1 *The fixed points of* $g(z)$ *are given by* $h : \mathbb{C} \mapsto \mathbb{C}$ *with*:

$$h(k, c) = \frac{W(k, -\log(c))}{-\log(c)}, \quad k \in \mathbb{Z}$$

Proof $z = g(z) \Leftrightarrow z = c^z \Leftrightarrow z\mathrm{e}^{-z\log(c)} = 1 \Leftrightarrow -z\log(c)\mathrm{e}^{-z\log(c)} = -\log(c) \Leftrightarrow m(-z\log(c)) = -\log(c) \Leftrightarrow -z\log(c) = W(k, -\log(c))$, $k \in \mathbb{Z}$, by

Definition 2.1 $\Leftrightarrow z = W(k, -\log(c))/-\log(c), k \in \mathbb{Z}$, and the lemma follows. ■

Lemma 3.2 *If* $c \in \mathbb{C} \setminus \{\mathrm{e}^{\mathrm{e}^{-1}}\}$ *and* $k \in \mathbb{Z} \setminus \{0\}$, *then* $h(k, c)$ *is a repeller*.

Proof $g'(h(k, c)) = \log(c)[W(k, -\log(c))/-\log(c)] = -W(k, -\log(c)) \in -D_k$, by Lemma 2.4, so if $k \in \mathbb{Z} \setminus \{0\}$, then $\left|g'(h(k, c))\right| > 1$, by Lemma 2.15, and the lemma follows. ■

The assumption $c \neq \mathrm{e}^{\mathrm{e}^{-1}}$ is crucial. Otherwise $g'(h(k, c)) = -W\left(k, -\mathrm{e}^{-1}\right) = 1$, for $k = -1$. (Lemma 2.14).

Lemmas 3.1 and 3.2 lead to the central lemma of this article.

Lemma 3.3 *Whenever* ${}^{\infty}c$ *exists finitely, its value is given by*:

$$h(c) = \frac{W(-\log(c))}{-\log(c)}.$$

4. CONVERGENCE FOR $c \in \mathbb{R}$

The fact that successive power iterates of $\sqrt{2}$ converge to 2 is numerically verified in Crandall [18, pp. 70, 74] and Israel [23, p. 66], and analytically explained in Spivak [40, p. 434], Länger [26, p. 77] and Mitchelmore [31, pp. 643–646]. Consider the following relation for the positive square root of 2. $(\sqrt{2})^2 = 2$. Replace the exponent with the equation's left side to get, $(\sqrt{2})^{(\sqrt{2})^2} = 2$. By repeating the process of this substitution of the last exponent recursively we obtain finite sequences of n radicals, $(\sqrt{2})^{(\sqrt{2})^{\cdots(\sqrt{2})^2}} = 2$, which are valid for all $n \in \mathbb{N}$. We have good reason to suspect that *informally*

$$^{\infty}(\sqrt{2}) = 2. \tag{4.1}$$

We shall prove this in Lemma 4.9.

LEMMA 4.1 *If* $x \in [\mathrm{e}^{-1}, \mathrm{e}]$, *then* $h(x^{x^{-1}}) = x$.

Proof By definition $h(y)$ solves the equation $x^{x^{-1}} = y$, so it is a partial local inverse of the function $y(x) = x^{x^{-1}}$. On the indicated interval $y(x)$ is 1–1 and onto the range of $[\mathrm{e}^{-1}, \mathrm{e}]$, and the lemma follows. ■

LEMMA 4.2 *If* $x \in (e, +\infty]$, *then* $h(x^{x^{-1}}) = w \in (1, e)$, *with* $w^{w^{-1}} = x^{x^{-1}}$.

Proof $y(x)$ is continuous on $(1, +\infty)$, attains a max at $x = e$, and $\lim_{x\to\infty} y(x) = 1$; so there exists a w in $(1, e)$, such that $w^{w^{-1}} = x^{x^{-1}}$, and the lemma follows from Lemma 4.1. ■

Whenever $y \in (1, \mathrm{e}^{\mathrm{e}^{-1}})$, the two values w, x which satisfy $w^{w^{-1}} = x^{x^{-1}} = y$ are always given as $\{h(-1, y), h(y)\}$. If $y \in (1, \mathrm{e}^{\mathrm{e}^{-1}})$, then $-\mathrm{e}^{-1} < -\ln(y) < 0$, so $\{W(-1, -\ln(y)), W(-\ln(y))\} \in \mathbb{R}$, by Lemmas 2.11, 2.12, and therefore $\{h(-1, y), h(y)\} \in \mathbb{R}$. If $y = \mathrm{e}^{\mathrm{e}^{-1}}$, then $h(-1, y) = h(y) = \mathrm{e}$, by Lemma 2.14.

Example $y = 1.3304 \doteq 1.562^{(1/1.562)} \doteq 6.620^{(1/6.620)}$. Such values are given here analytically by W, but are also found by numerical or other methods in the articles which deal with solving the equation $x^y = y^x$, Bush [12, p. 763], Carmichael [13, pp. 222–226], [14, pp. 78–83], Franklin [21, p. 137], Moulton [32, pp. 233–237], Sato [36, p. 316], Slobin [39, pp. 444–447], Archibald [1, p. 141], and Kupitz [22, pp. 96–99]. The graph of $y(x)$ is also shown in Knoebel [24, p. 236]. Knoebel observes that $y(x)$ is a partial inverse of $h(x)$, although he does not explicitly define h via W.

Lemmas 4.1 and 4.2 summarized.

LEMMA 4.3

$$h\left(x^{x^{-1}}\right) = \begin{cases} x, & \textit{if } x \in [\mathrm{e}^{-1}, \mathrm{e}]; \\ w, & w \in (1, \mathrm{e}) : w^{w^{-1}} = x^{x^{-1}}, \quad \textit{if } x \in (\mathrm{e}, +\infty). \end{cases}$$

The interval of convergence for the real case can now be determined from fixed point iteration. The only potentially attractive fixed point of g(x) is given by Lemma 3.3, as h(c). Using elementary properties of the functions involved, if $|g'(h(c))| \le 1$, then $|-W(-\ln(c))| \le 1$. This means $W(-\ln(c)) \in [-1, 1]$, thus $m(W(-\ln(c))) \in m([-1, 1])$, or $m(W(-\ln(c))) \in [-e^{-1}, \mathrm{e}]$, so $-\ln(c) \in [-e^{-1}, \mathrm{e}]$, by Definition 2.2, and finally $c \in [\mathrm{e}^{-\mathrm{e}}, \mathrm{e}^{\mathrm{e}^{-1}}]$.

LEMMA 4.4 *If* $c = \mathrm{e}^{-\mathrm{e}}$, $x_0 = \ln\big(W\big(\ln(c)^{-1}\big)\ln(c)^{-1}\big)\ln(c)^{-1}$, *and* $u(x) = g^{(2)}(x) - x$, *then,*

$$x_0 \text{ is the only critical point of } u(x), \text{ in } [0,1] \tag{4.2a}$$

$$x_0 = e^{-1} \tag{4.2b}$$

$$u(x_0) = 0 \tag{4.2c}$$

$$\left(\frac{du}{dx}\right)_{x_0} = 0 \tag{4.2d}$$

$$\frac{du}{dx} < 0, x \in [0,1] - \{x_0\}. \tag{4.2e}$$

Proof $du/dx = 0$ can be solved analytically using W. If $du/dx = 0$, then $g^{(2)}(x)g(x)\ln(c)^2 = 1$, so $\mathrm{e}^{y\ln(c)}y\ln(c) = \ln(c)^{-1}$. If $y = c^x$, then $m(y\ln(c)) = \ln(c)^{-1}$, so $y\ln(c) = W\big(k, \ln(c)^{-1}\big)$; therefore $y = W\big(k, \ln(c)^{-1}\big)\ln(c)^{-1}$, and finally $x = \ln\big(W\big(k, \ln(c)^{-1}\big)\ln(c)^{-1}\big)\ln(c)^{-1}$, $k \in \mathbb{Z}$. Equation (4.2a) follows immediately from Lemmas 2.10 and 2.14. Equation (4.2b) follows immediately from Lemma 2.14 and algebra. Equation (4.2c) then follows trivially from (4.2b). Equation (4.2d) also follows trivially. For (4.2e) note that $\ln(c) = -\mathrm{e} < 0$, so, if $x < x_0$ then $g(x) > \mathrm{e}^{-1}$, and $g^{(2)}(x) < \mathrm{e}^{-1}$, so $g^{(2)}(x)g(x)\ln(c)^2 < 1$, consequently $du/dx < 0$. For $x > x_0$ the proof (with inequalities reversed) is similar and the lemma follows. ■

LEMMA 4.5 *If* $c \in \{\mathrm{e}^{-\mathrm{e}}, \mathrm{e}^{\mathrm{e}^{-1}}\}$, *then* ${}^{\infty}c = h(c)$.

Proof If $c = \mathrm{e}^{-\mathrm{e}}$, the fixed point of the function $g(x)$ is given by Lemma 3.3. $h(c) = h(\mathrm{e}^{-\mathrm{e}}) = W(-\ln(\mathrm{e}^{-\mathrm{e}}))/-\ln(\mathrm{e}^{-\mathrm{e}}) = W(e)/e = \mathrm{e}^{-1}$, by Lemma 2.13. Using Lemma 4.4, continuity of $u(x)$, and the facts: $u(0) = c > 0$, $u(1) = c^c - 1 < 0$, it follows that $g^{(2)}(x) > x$, if $x \in [0, \mathrm{e}^{-1})$ and $g^{(2)}(x) < x$, if $x \in (\mathrm{e}^{-1}, 1]$. Using the last two inequalities and induction on n, the sequence: $a_n = g^{(n)}(1)$, $n \in \mathbb{N}$ satisfies, $a_{2n+2} < a_{2n}$, and $a_{2n+3} > a_{2n+1}$, for all $n \in \mathbb{N}$. The latter show that a_{2n+1} and a_{2n} are monotone increasing and monotone decreasing respectively. Further, since $0 < c = \mathrm{e}^{-\mathrm{e}} < 1$, both sequences are bounded above by 1 and below by 0. It follows that a_{2n+1} and a_{2n} both possess limits. Since the only single root of $u(x)$ is x_0 (otherwise (4.2a) is violated), both sequences must converge to x_0, from which follows that a_n converges to $x_0 = \mathrm{e}^{-1}$.

If $c = \mathrm{e}^{\mathrm{e}^{-1}}$, the fixed point of the function $g(x)$ is given again by Lemma 3.3. $h(c) = h(\mathrm{e}^{\mathrm{e}^{-1}}) = W(-\ln(\mathrm{e}^{\mathrm{e}^{-1}}))/-\ln(\mathrm{e}^{\mathrm{e}^{-1}}) = W(-\mathrm{e}^{-1})/-\mathrm{e}^{-1} = e$, by Lemma 2.14. By induction on n the sequence $a_n = g^{(n)}(1)$, $n \in \mathbb{N}$ is strictly increasing and bounded above by e, so it converges to e and the lemma follows. ■

Knoebel [24, p. 240], Barrow [7, p. 153], and De Villiers and Robinson [42, pp. 14–15] arrive at the same result differently, without using the W function.

LEMMA 4.6 *If* $c \in (0, \mathrm{e}^{-\mathrm{e}})$, *then* ${}^{\infty}c$ *does not exist.*

Proof The fixed point $h(c)$ of $g(x)$ from Lemma 3.3 is a repeller. If $c \in (0, \mathrm{e}^{-\mathrm{e}})$, then $W(e) < W(-\ln(c))$ by Lemma 2.11 and thus $1 < W(-\ln(c))$ by Lemma 2.13. This means $1 < |g'(h(c))|$, and the lemma follows from fixed point iteration. ■

LEMMA 4.7 *If* $c \in (\mathrm{e}^{-\mathrm{e}}, \mathrm{e}^{\mathrm{e}^{-1}})$, *then* ${}^{\infty}c = h(c)$.

Proof The fixed point $h(c)$ of $g(x)$ from Lemma 3.3 is an attractor. If $c \in (\mathrm{e}^{-\mathrm{e}}, \mathrm{e}^{\mathrm{e}^{-1}})$, then $-\mathrm{e}^{-1} < -ln(c) < \mathrm{e}$, so $W(-\mathrm{e}^{-1}) < W(-\ln(c)) < W(e)$ by Lemma 2.11, and thus $|g'(h(c))| < 1$ by Lemmas 2.13 and 2.14, and the lemma follows from fixed point iteration. ■

LEMMA 4.8 *If* $c \in (\mathrm{e}^{\mathrm{e}^{-1}}, +\infty)$, *then* ${}^{\infty}c$ *does not exist.*

Proof The fixed point $h(c)$ of $g(x)$ from Lemma 3.3 is a repeller. If $c \in (\mathrm{e}^{\mathrm{e}^{-1}}, +\infty)$, then $-\ln(c) \in BC_0$; so $W(-\ln(c)) \in C_0$ by Lemma 2.3, and therefore $|g'(h(c))| > 1$ by Lemma 2.15, and the lemma follows from fixed point iteration. ■

Lemmas 4.5–4.8 establish the final lemma of this section.

LEMMA 4.9 *If* $c \in [\mathrm{e}^{-\mathrm{e}}, \mathrm{e}^{\mathrm{e}^{-1}}]$, *then* ${}^{\infty}c = h(c)$

Using Lemma 4.9 for $c = \mathrm{e}^{\mathrm{e}^{-1}}$, $c = \mathrm{e}^{-\mathrm{e}}$, and $c = \sqrt{2}$,

$$
{}^{\infty}\left(\mathrm{e}^{\mathrm{e}^{-1}}\right) = \left(\mathrm{e}^{\mathrm{e}^{-1}}\right)^{\left(\mathrm{e}^{\mathrm{e}^{-1}}\right)^{\cdots}} = h\left(\mathrm{e}^{\mathrm{e}^{-1}}\right) = \mathrm{e}
$$

$$
{}^{\infty}(\mathrm{e}^{-\mathrm{e}}) = (\mathrm{e}^{-\mathrm{e}})^{(\mathrm{e}^{-\mathrm{e}})^{\cdots}} = h(\mathrm{e}^{-\mathrm{e}}) = \mathrm{e}^{-1}
$$

$$
{}^{\infty}\left(\sqrt{2}\right) = (\sqrt{2})^{(\sqrt{2})^{\cdots}} = h(2^{1/2}) = 2
$$

The last equation settles the question posed in the beginning of this section with Eq. (4.1). That an algebraic infinite exponential converges if and only if its base belongs to the interval $[\mathrm{e}^{-\mathrm{e}}, \mathrm{e}^{\mathrm{e}^{-1}}] \doteq [0.06598, 1.44466]$ is also established in Knoebel [24, p. 240], Mitchelmore [31, p. 645], and Ogilvy [33, p. 556] using other methods, without using the W function.

Ash [2, pp. 207–208] and Macdonnell [28, pp. 301–303] establish that for $k \in \mathbb{N}$, $\lim_{c \to 0^+} {}^{2k}c = 1$ and $\lim_{c \to 0^+} {}^{2k+1}c = 0$. Whenever $c \in (0, \mathrm{e}^{-\mathrm{e}})$, $\{{}^{n}c\}_{n \in \mathbb{N}}$ is a 2-cycle, by considering the even and odd subsequences, ${}^{2n}c$ and ${}^{2n+1}c$. The bifurcation which occurs and its behavior and properties are analyzed in Ash [2, p. 207], De Villiers and Robinson [42, p. 15] and Macdonnell [28, p. 299]. We note that the two branches stemming from the bifurcation point $\{\mathrm{e}^{-\mathrm{e}}, \mathrm{e}^{-1}\}$ can be parametrized as $a^{a/(1-a)}$ and $a^{1/(1-a)}$ for appropriate positive a (see for example Knoebel [24, p. 237] or Voles [43, p. 212]). In this case, as shown in Spivak [40, p. 434], Knoebel [24, pp. 241–243], De Villiers and Robinson [42, p. 13] and Lense [27, p. 501], the two separate limits $a = \lim_{n \to \infty} {}^{2n+1}c$ and $b = \lim_{n \to \infty} {}^{2n}c$ satisfy $0 < a < h(c) < b < 1$ and the *second auxiliary equation system*,

$$
\begin{Bmatrix} a = c^{c^{a}} \\ b = c^{a} \end{Bmatrix}. \tag{4.3}
$$

An analytic solution to system (4.3) will be presented in a forthcoming article [22], in which we will present the difficulties of solving the nth auxiliary equation for the complex function $g(z)$ using a function similar to W.

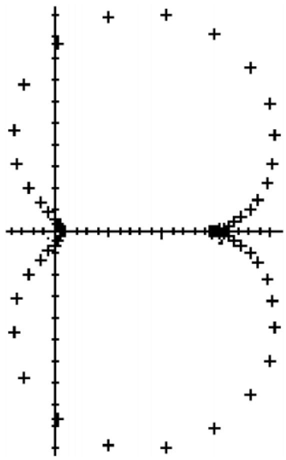

FIGURE 3 $\{\phi(t) : t^{60} = 1\}$.

5. CONVERGENCE FOR $c \in \mathbb{C}$

Let D be the unit disk and consider the map $\phi: \mathbb{C} \mapsto \mathbb{C}$, defined as: $\phi(z) = e^{(z/e^z)} = e^{-m(-z)}$. The image of D under ϕ is a certain nephroid region $N = \phi(D)$. This region is shown in Fig. 3.

Shell in [38, p. 679], [37, p. 12], and Baker and Rippon in [5, p. 106] show that inside N we have convergence.

Theorem 5.1 (*Shell*) *$\{^n(\phi(t))\}_{n\in\mathbb{N}}$ converges to e^t in some neighborhood of e^t if $|t| < 1$ and can do so only if $|t| \le 1$.*

Shell does not address the question of what happens on ∂D. This is addressed by Baker and Rippon in [4, p. 502] and settled in [3]:

Theorem 5.2 (*Baker and Rippon*) *$\{^n c\}_{n\in\mathbb{N}}$ converges for $\log(c) \in \{te^{-t}: |t| < 1$, or $t^n = 1$, for some $n \in \mathbb{N}\}$ and it diverges elsewhere.*

t and c are related via W, considering (as usual) the principal branches of the functions involved. $c = \phi(t) \Leftrightarrow W(-\log(c)) = W(m(-t)) \Leftrightarrow t = -W(-\log(c))$ using Corollary 2.7 and Lemma 2.15. Therefore $\phi^{-1} = (-W) \circ (-\log)$ and $t = \phi^{-1}(c)$. In view of the last equation Theorem 5.2 becomes,

Theorem 5.3 *$\{^n c\}_{n\in\mathbb{N}}$ converges if $\left|\phi^{-1}(c)\right| < 1$ or $\left(\phi^{-1}(c)\right)^n = 1$, for some $n \in \mathbb{N}$ and it diverges elsewhere.*

Lemma 5.4 *If $c \in \mathbb{C}$, then the multiplier of the fixed point (c) of $g(z)$ is $t = \phi^{-1}(c)$.*

Proof $g'(h(c)) = \log(c)[W(-\log(c))/(-\log(c))] = -W(-\log(c)) = t$, and the lemma follows. ■

Theorem 5.2 then alternatively states that if $c = \phi(t)$, then $\{g^{(n)}(c)\}_{n\in\mathbb{N}}$ converges only if the modulus of the multiplier $t = \phi^{-1}(c)$ of the fixed point $h(c)$ of $g(z)$ is less than one, or if the multiplier is an nth root of unity.

Lemma 5.5 *If $|t| < 1$ and $c = \phi(t)$, then $^\infty c = h(c)$.*

Proof $g(z)$'s fixed point is $h(c)$ by Lemma 3.3.

$$h(c) = \frac{W(-\log(c))}{-\log(c)} = \frac{W(-te^{-t})}{-te^{-t}} = \frac{W(m(-t))}{-te^{-t}} = \frac{-t}{-te^{-t}} = e^t,$$

using Lemma 2.15, and Corollary 2.7. The fixed point $h(c) = e^t$ of $g(z)$ is an attractor. By Lemma 5.4, $|g'(h(c))| = |t| < 1$, which is true by the hypothesis and the lemma follows from fixed point iteration or Theorem 5.1 and Lemma 3.3. ■

LEMMA 5.6 *If* $|t| > 1$ *and* $c = \phi(t)$, *then* ${}^{\infty}c$ *does not exist.*

Proof The fixed point $h(c) = e^t$ of $g(z)$ is a repeller. By Lemma 5.4, $|g'(h(c))| = |t| > 1$, which is true by the hypothesis and the lemma follows from fixed point iteration or Theorem 5.1. ■

We note that 'divergence' here does not necessarily mean 'proper divergence' (i.e., ${}^{\infty}c = \infty$). Rather, it means either proper divergence or periodic cycling. Periodic points given by Lemma 3.1 have a period 1, so it trivially follows that these points are automatically solutions of the nth auxiliary equation $g^{(n)}(z) = z$, $n \in \mathbb{N}$, but not all solutions of the latter are given by Lemma 3.1. Such points exist. Shell in [37, p. 28] makes implicit use of the following for the multiplier t of a point z:

$$\left(g^{(n)}\right)'(z) = [\log(c)]^n \prod_{k=1}^{n} g^{(k)}(z). \tag{5.1}$$

Using (5.1) (which can be shown using induction), it is straightforward to numerically verify that for $c = 3 - 3i \notin N$, the points $\{0, 1, {}^{1}c, {}^{2}c, {}^{3}c\}$ are all periodic points of $g^{(5)}(z)$ with primitive period 5, since for any such z_0, $\left|\left(g^{(5)}\right)'(z_0)\right| = 0$, meaning that $\{g^{(n)}(z_0)\}_{n\in\mathbb{N}}$ is a super-attracting 5-cycle.

The fixed point condition fails with $t^n = 1$. By Lemma 5.4 $|g_k'(h(c_k))| = |t_k| = 1$, since all the nth roots t_k lie on the unit circle. The fixed point condition also fails on the boundary ∂D for periods greater than 1. Using (5.1) and Lemma 5.4 $|\left(g^{(n)}\right)'(h(c_k))| = \log(c_k)^n (h(c_k))^n = \left|\log(c_k)^n (-W(-\log(c_k)))^n / \log(c_k)\right|^n = \left|t_k^n\right| = 1$, and this does not give us any further information. What happens on the boundary falls into two different cases:

$$|t^n| = 1 \Leftrightarrow (|t| = 1 \text{ and } t^n = 1) \quad \text{or} \quad (|t| = 1 \text{ and } t^n \neq 1),\ n \in \mathbb{N}.$$

LEMMA 5.7 *If* $|t| = 1$, $t^n = 1$, and $c = \phi(t)$, *then* ${}^{\infty}c = h(c)$.

Proof Let $t_k = e^{(2k\pi/n)i}$ be a complex nth root of unity, $k \in \{0, 1, \ldots, n-1\}$ and consider the functions $g_k(z) = (c_k)^z$, with $c_k = \phi(t_k)$. The fixed points of $g_k(z)$ are $h(c_k)$, by Lemma 3.3.

$$h(c_k) = \frac{W(-\log(c_k))}{-\log(c_k)} = \frac{W(m(-t_k))}{-t_k e^{-t_k}} = \frac{-t_k}{-t_k e^{-t_k}} = e^{t_k}$$

using Lemma 2.15 and Corollary 2.7, and the lemma follows from Theorem 5.2 and Lemma 3.3. ■

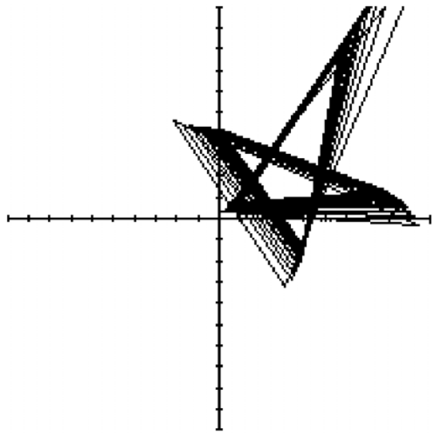

FIGURE 4 Trajectory for $t = e^{(4\pi i/5)}$ perturbed.

Since $\phi(\partial D) = \partial N$, the unit circle gets mapped onto the boundary of N under ϕ of Lemma 5.5 or onto a cardioid, under the map $-m(-z) = \log(\phi)$. In particular, $\phi(-1) = e^{-e}$, $\phi(1) = e^{e^{-1}}$, $\phi(i) \doteq 1.98933 + 1.19328i$, and $\phi(-i) = \overline{\phi(i)}$. N is established and drawn in Thron [41, p. 741], Shell [37, p. 28], and Baker and Rippon [4].

We note (as in Branner [11, p. 40]) that the fixed points $h(c_k) = e^{t_k}$ of the functions $g_k(z)$ are rationally indifferent or parabolic. Their multiplier is exactly $t_k = e^{2\pi i\alpha}$, $\alpha \in \mathbb{Q}$, thus they are not linearizable. They are also dense in ∂D. If we perturb z away from e^{t_k}, the iterates of $g_k(z)$ will eventually form a p-cycle where $p = n/GCD(n,k)$. i.e., the sequence $\{g_k^{(n)}(z)\}_{n\in\mathbb{N}}$ will eventually stabilize onto the cycle $\{g_k^{(n)}(c_0)\}$, $n \in \{1, 2, \ldots, p\}$ where c_0 satisfies the pth auxiliary equation $g_k^{(p)}(z) = z$, but no auxiliary equation $g_k^{(n)}(z) = z$, $n < p$. If we iterate $g_k(z)$ unperturbed, the p-cycle will eventually coalesce into a 1-cycle.

Figure 4 displays the 5-cycle trajectory of the iteration for $n = 5$, $k = 2$, perturbed 3.5 away from e^{t_k}.

6. FRACTALS RELATED TO INFINITE EXPONENTIALS

The limit of many complex infinite exponentials can now be found by Lemma 3.3, provided we know that the corresponding infinite exponential converges. What is ${}^{\infty}i$ (if anything)? Macintyre [29, p. 67] without using W establishes that ${}^{\infty}i$ converges. On the other hand, using $g(z)$ with $c = i$, by Lemma 5.4 the multiplier of the fixed point $h(i)$ is $t = -W(-\pi i/2)$ whose modulus can be evaluated numerically to less than 1. Note also that $i \in N$, so $|\phi^{-1}(i)| < 1$ (Fig. 3). Therefore $\{{}^{n}i\}_{n\in\mathbb{N}}$ by the Central Lemma 3.3 and fixed point iteration or Theorem 5.1 converges to:

$$ {}^{\infty}i = h(i) = \frac{2iW(-\pi i/2)}{\pi} \doteq 0.43828 + 0.36059i. \tag{6.1} $$

It is expected now to ask if infinite exponentiation can produce an interesting fractal. Indeed, it does produce beauty, since complex numbers are involved and the domain

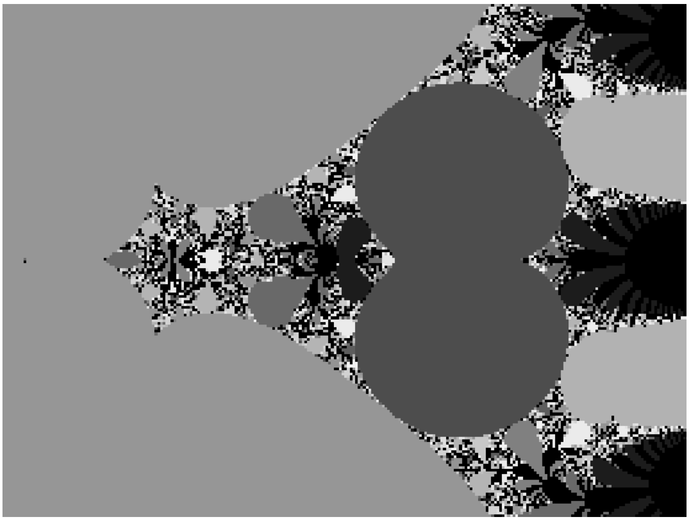

FIGURE 5 Parameter map for $f(z) = c^z$.

of convergence of infinite exponentials can be extended to such numbers. Figure 5 is the parameter space (or Mandelbrot set) for the exponential map $g(z) = c^z$, while Fig. 6 is the Julia set for $c = i$.

The red nephroid bulb in Fig. 5 is the region N given by Baker and Rippon in [3] and [5] with Theorem 5.1. This is the basin for points of period 1. For points inside this basin, $\{{}^n z\}_{n \in \mathbb{N}}$ always converges to $h(z)$ by the central Lemma 3.3. We note the connection with Fig. 3. This basin is also sketched in Baker and Rippon using the map $\log(\phi) = te^{-t}$ instead. The rest of the domains are domains for periods greater than 1. The little yellow circle-like domain just left of domain N, is period 2. The two green domains on the upper and lower left are period 3. The black pixels in Fig. 5 are unknown areas of connected Fatou components where the machine was unable to determine whether there was escape for some reason, such as insufficient or low iterations. It is not known whether there are finitely or infinitely many domains of period $k \geq 3$.

Figure 5 was drawn using a *Mandelbrot* process. We can instead alter the process to produce *Julia* sets as well by fixing c and iterating $g(z)$ with $z \in \mathbb{C}$ (see for example Becker and Dörfler [8, p. 113]).

Figure 6 is the Julia set for $c = i$, $J(i^z)$. The main circular feature is the attractor given by (6.1) which is located at $h(i)$. This point is linearizable by the Kœnings Theorem (see Branner [11, p. 40]). Its multiplier satisfies $|t| < 1$ and therefore there is a neighborhood U of $h(i)$, where $g(U) \subset U$. If we set $\sigma_n(z) = g^{(n)}(z)/t^n$, then the holomorphic functions σ_n converge in U uniformly on compact subsets to a holomorphic map σ. Furthermore in view of equations (3.1) and (5.1),

$$\sigma'(h(i)) = \lim_{n \to \infty} \frac{\sigma_n'(h(i))}{t^n} = \lim_{n \to \infty} \frac{\log(i)^n h(i)^n}{t^n} = 1,$$

so σ provides the required local coordinate change. The secondary circular features are a continuum of repellers.

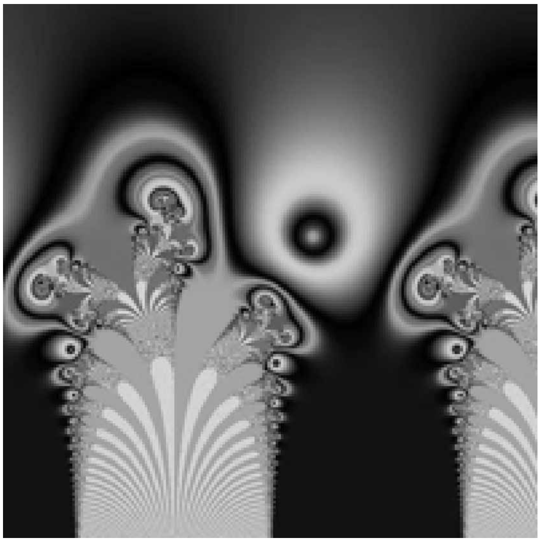

FIGURE 6 $J(i^z)$.

Since $i \in N$, this particular Julia set contains no periodic points of period $n > 1$. The basin of attraction (the complement of $J(i^z)$, or Fatou set $F(i^z)$) is $\mathbb{C} \setminus J(i^z)$ and is shown in darker shades. $F(i^z)$ in this case falls under case (1) in the Sullivan classification Theorem (see Bergweiler [10, pp. 12–13]). $J(i^z)$ is a *Cantor bouquet* (see for example Peitgen and Richter [34, p. 33] and Devaney [20, p. 3], [19, pp. 1–4]).

Incidentally, the fractal in Fig. 5 is very similar to the fractal for the iterates of $g(z)$ when $c = e$ (the exp function) shown in Peitgen and Richter [34, p. 34]. In this case, W again plays a crucial role since the fixed points of $g(z)$ are also given by Lemma 3.1 as $c_k = -W(k, -1)$, $k \in \mathbb{Z}$. We note that $g(z)$ has infinitely many fixed points and no fixed points of $g(z)$ are attractors. $|g'(c_k)| = |\mathrm{e}^{c_k}| = |c_k| = |W(k, -1)| > 1$ by Lemma 2.3, since for all $k \in \mathbb{Z}$, $W(k, -1) \in BC_k$ (see the proof of Lemma 3.2).

7. THE ANALYTIC CONTINUATION OF h

We now extend h to complex z, as follows:

$$h(k, l, z) = \frac{-W(k, -\log(l, z))}{-\log(l, z)}, k, l \in \mathbb{Z}. \tag{7.1}$$

The definition is unambiguous, provided we specify the branches of W, and log via k and l, so then all involved functions are single-valued and $h(k, l, z)$ is then well-defined. $h(z)$ has two branch points and two branch cuts. The first branch point is at 0 and the first branch cut that is the negative real axis (because of the *log*), and there is a second branch point at $\mathrm{e}^{\mathrm{e}^{-1}}$, with a branch cut that is the subset of the positive real axis from $\mathrm{e}^{\mathrm{e}^{-1}}$ to infinity. To find the second branch point z, we note that since $-\mathrm{e}^{-1}$ is a branch point of the W, z has to satisfy: $-\log(z) = -\mathrm{e}^{-1}$, so $z = \mathrm{e}^{\mathrm{e}^{-1}}$. The value of h

there is $h(\mathrm{e}^{\mathrm{e}^{-1}}) = e$. A power series for h can now be derived directly using the properties of W. Specifically, the principal branch of log, is analytic everywhere except on the negative real axis, including 0, where log is not even defined. The principal branch of the W, on the other hand, is analytic at 0, with a radius of convergence $R_S = \mathrm{e}^{-1}$. We expect therefore the resultant composition (7.1) to be analytic at least in some region D_h (which, of course, excludes its two branch cuts, namely: $(-\infty, 0) \cup (\mathrm{e}^{\mathrm{e}^{-1}}, +\infty)$).

LEMMA 7.1 *$h(z)$ is analytic in $D_S = \{z\colon |\log(z)| \le \mathrm{e}^{-1}\}$ with series expansion*:

$$S(z) = \sum_{n=1}^{\infty} \frac{(n\log(\mathrm{z}))^{n-1}}{n!}.$$

Proof Set $Z = -\log(z)$ in the series given in Lemma 2.16 and then divide by $-\log(z)$. The Ratio Test reveals the radius of convergence. $\lim_{n\to\infty} |a_{n+1}/a_n| = \lim_{n\to\infty} |(1 + 1/n)^{n-1} \log(z)| = |e\log(z)| < 1$, or equivalently $|\log(z)| < \mathrm{e}^{-1}$, and $D_S = \{z\colon |\log(z)\} < \mathrm{e}^{-1}|$ is established. Using Stirling's approximation when $|\log(z)| = \mathrm{e}^{-1}$ (in a similar way as in Lemma 2.16) we obtain the sharper estimate, $D_S = \{z\colon |\log(\mathrm{z})| \le \mathrm{e}^{-1}\}$. ∎

This series is given by Knoebel in [24, p. 244] (quoting papers by Eisenstein and Wittstein), and Barrow in [7, p. 159], although neither author uses W. For the real case, Knoebel notes that the series is valid for $\mathrm{e}^{-\mathrm{e}^{-1}} < x < \mathrm{e}^{\mathrm{e}^{-1}}$. We note that $h(z)$ fails to be analytic past the two points $\mathrm{e}^{-\mathrm{e}^{-1}}$ and $\mathrm{e}^{\mathrm{e}^{-1}}$ as predicted by Knoebel. It is also interesting to note that D_S is nothing more than the region of analyticity of the principal branch of W under the map exp.

A quick application of Lemma 7.1 and Lemma 4.3 gives,

$$\sum_{n=1}^{\infty} \frac{n^{n-1}}{\mathrm{e}^{n-1} n!} = e$$

$$\sum_{n=1}^{\infty} \frac{(n \ln(2))^{n-1}}{2^{n-1} n!} = 2$$

$$\sum_{n=1}^{\infty} \frac{(n \ln(x))^{n-1}}{x^{n-1} n!} = h\left(x^{x^{-1}}\right), \quad \left|\frac{\ln(x)}{x}\right| \le e^{-1}$$

We finish this section with some more properties of h. It is real valued for real values of z in the interval $[\mathrm{e}^{-\mathrm{e}}, \mathrm{e}^{\mathrm{e}^{-1}}]$, since we have established that on this interval limit 1.2 exists finitely and is given by Lemma 3.3. On the interval $(0, \mathrm{e}^{-\mathrm{e}})$, $h(z)$ simply provides for the repelling fixed point of Lemma 4.6. We finally note that an application of Lemma 7.1 for $z = \mathrm{e}^w$ gives

$${}^{\infty}(\mathrm{e}^w) = h(\mathrm{e}^w) = \frac{W(-w)}{-w}$$

which is analytic in the region $D'_S = \{z\colon |z| \le \mathrm{e}^{-1}\}$, even though all finite iterates are entire. We take a look at the exact expansions for finite and infinite iterates of e^z and z in Section 9.

8. THE EXPANSION OF $exp(z\sum_{n=0}^{\infty} a_n z^n)$

LEMMA 8.1 *If* $s(z) = \sum_{n=0}^{\infty} a_n z^n$ *and* $\mathrm{e}^{zs(z)} = \sum_{n=0}^{\infty} b_n z^n$, *then*

$$b_n = \begin{cases} 1, & \text{if } n = 0, \\ \dfrac{\sum_{j=1}^{n} j b_{n-j} a_{j-1}}{n}, & \text{otherwise} \end{cases}.$$

Proof Disregarding issues of convergence, if $t(z) = \mathrm{e}^{zs(z)}$, then $t'(z) = (zs(z))'\mathrm{e}^{zs(z)} = \left(\sum_{n=0}^{\infty} a_n z^{n+1}\right)' t(z) = t(z)\sum_{n=0}^{\infty}(n+1)a_n z^n$, and the lemma follows from equating coefficients between the corresponding expansions and changing indexes. ∎

A simple inductive argument on m, shows that $a_{m,0} = 1$, for all $m \in \mathbb{N}$, so in view of Definition 1.4 and Lemma 8.1 the recursive relationship for the coefficients of the iterates is now complete.

$$a_{m,n} = \begin{cases} 1, & \text{if } n = 0, \\ \dfrac{1}{n!}, & \text{if } m = 1, \\ \dfrac{\sum_{j=1}^{n} j a_{m,n-j} a_{m-1,j-1}}{n}, & \text{otherwise} \end{cases}. \tag{8.1}$$

Equation 8.1 does not reveal any discernible patterns, until one tabulates the values (Table I).

9. THE RE-EMERGENCE OF W

LEMMA 9.1 *If* $m, n \in \mathbb{N}$ *and* $m \geq n$, *then* $a_{m,n} = a_{n,n}$.

Proof We use induction on n. It is clear that $a_{1,1} = 1$ and $a_{m,1} = 1$, for all $m \in \mathbb{N}$. If $a_{k,1} = 1$, then $a_{k+1,1} = \sum_{j=1}^{1} j a_{k+1,1-j} a_{k,j-1} = a_{k+1,0} a_{k,0} = 1 = a_{1,1}$, therefore $a_{m,1} = a_{1,1}$, for all $m \in \mathbb{N}$.

TABLE I $a_{m,n}$ for ${}^m(\mathrm{e}^z)$, $(m,n) \in \{1,2,\ldots,6\} \times \{0,1,\ldots,6\}$

m:n	*0*	*1*	*2*	*3*	*4*	*5*	*6*
1	1	1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{24}$	$\frac{1}{120}$	$\frac{1}{720}$
2	1	1	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{41}{24}$	$\frac{49}{30}$	$\frac{1057}{720}$
3	1	1	$\frac{3}{2}$	$\frac{8}{3}$	$\frac{101}{24}$	$\frac{63}{10}$	$\frac{6607}{720}$
4	1	1	$\frac{3}{2}$	$\frac{8}{3}$	$\frac{125}{24}$	$\frac{49}{5}$	$\frac{12847}{720}$
5	1	1	$\frac{3}{2}$	$\frac{8}{3}$	$\frac{125}{24}$	$\frac{54}{5}$	$\frac{16087}{720}$
6	1	1	$\frac{3}{2}$	$\frac{8}{3}$	$\frac{125}{24}$	$\frac{54}{5}$	$\frac{16807}{720}$

Now assume $a_{m,k} = a_{k,k}$ is true for all $m \geq k$. We have to show that if $m \geq k+1$, then $a_{m,k+1} = a_{k+1,k+1}$, or equivalently using (8.1),

$$\frac{\sum_{j=1}^{k+1} j a_{m,k+1-j} a_{m-1,j-1}}{k+1} = \frac{\sum_{j=1}^{k+1} j a_{k+1,k+1-j} a_{k,j-1}}{k+1}. \tag{9.1}$$

If $m \geq k+1$, then $m \geq k+1-j$, so that by the induction hypothesis,

$$a_{m,k+1-j} = a_{k+1-j,k+1-j}. \tag{9.2}$$

Also, $k+1 \geq k+1-j$, therefore again by the induction hypothesis,

$$a_{k+1,k+1-j} = a_{k+1-j,k+1-j} \tag{9.3}$$

Equations (9.2) and (9.3) establish,

$$a_{m,k+1-j} = a_{k+1,k+1-j} \tag{9.4}$$

On the other hand, $j \leq k+1$, and so $j-1 \leq k$ and $m \geq k+1$, so $m-1 \geq k$, therefore $j-1 \leq m-1$, thus by the induction hypothesis,

$$a_{m-1,j-1} = a_{j-1,j-1} \tag{9.5}$$

and finally, $j \leq k+1$, so $j-1 \leq k$, and therefore, by the induction hypothesis,

$$a_{k,j-1} = a_{j-1,j-1}. \tag{9.6}$$

Equations (9.5) and (9.6) establish,

$$a_{m-1,j-1} = a_{k,j-1} \tag{9.7}$$

and (9.1) follows from (9.4) and (9.7). ■

Lemma 9.1 shows that as m grows larger, the coefficients eventually 'stabilize'. To what though? The answer comes from Lemma 3.3.

Corollary 9.2 *Wherever* ${}^{\infty}(e^z)$ *exists*, ${}^{\infty}(e^z) = W(-z)/-z = e^{-W(-z)}$.

Therefore, using Lemma 2.16 and the Ratio Test,

Corollary 9.3 ${}^{\infty}(e^z)$ *is analytic at the origin, with series expansion:*

$${}^{\infty}(e^z) = \sum_{n=0}^{\infty} \frac{(n+1)^n}{(n+1)!} z^n$$

and radius of convergence: $R_s = e^{-1}$.

Using the expansion in Lemma 7.1, Lemma 9.1, and Corollary 9.3, we can handle the expansions for the finite iterates.

COROLLARY 9.4 *For $m \in \mathbb{N}$, ${}^{m}(\mathrm{e}^{z})$ is entire, with series expansion:*

$${}^{m}(\mathrm{e}^{z}) = \sum_{n=0}^{m} \frac{(n+1)^{n}}{(n+1)!} z^{n} + \sum_{n=m+1}^{\infty} a_{m,n} z^{n}$$

where $a_{m,n}$ are given by (8.1).

COROLLARY 9.5 *For $m \in \mathbb{N}$, ${}^{m}z$ is analytic for z in the domain of* log, *with series expansion:*

$${}^{m}z = \sum_{n=0}^{m} \frac{(n+1)^{n}}{(n+1)!} \log(z)^{n} + \sum_{n=m+1}^{\infty} a_{m,n} \log(z)^{n}$$

where $a_{m,n}$ are given by (8.1).

The expansions for ${}^{m}(\mathrm{e}^{z})$ are valid for all $z \in \mathbb{C}$ if m is finite, since ${}^{m}(\mathrm{e}^{z})$ is always a composition of entire functions, thus itself entire. If m is infinite, then the expansion is valid only for $z \in \left\{z\colon |z| \leq \mathrm{e}^{-1}\right\}$.

The expansions for ${}^{m}z$ are valid in $\mathbb{C} \setminus \{x \in \mathbb{R}\colon x \leq 0\}$, if m is finite. If m is infinite, then the expansion is valid only for $z \in \{z\colon \left|\log(z)\right| \leq \mathrm{e}^{-1}\}$. The latter domain, restricted to $\mathbb{R}$, is the domain of convergence for the expansion of ${}^{\infty}z$ given also by Knoebel [24, p. 244] (Knoebel does not address what happens at the endpoints) and Barrow [7, p. 159].

10. INCIDENTAL IDENTITIES INVOLVING W

The Corollaries in Section 9 provide for an inexhaustible engine for identities in the spirit of $\int_0^1 x^x dx = \sum_{n=1}^{\infty} (-1)^{n+1}/n^n$.

COROLLARY 10.1

$$\int_0^{\mathrm{e}^{-1}} W(x)\, dx = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{(n+1)!\mathrm{e}^{n+1}} = \frac{1}{2}.$$

Proof $\int W(x)dx = \sum_{n=1}^{\infty} (-n)^{n-1}/n! \int x^n dx = \sum_{n=1}^{\infty} (-n)^{n-1}/(n+1)! x^{n+1}$, whenever $|x| \leq \mathrm{e}^{-1}$ using Lemma 2.16, and the Lemma follows from evaluating the integral at $x = \mathrm{e}^{-1}$ and at $x = 0$. ■

Evaluating the integral of Corollary 10.1 at $x = \mathrm{e}^{-1}$ and at $x = -\mathrm{e}^{-1}$ instead,

COROLLARY 10.2

$$\int_{-\mathrm{e}^{-1}}^{\mathrm{e}^{-1}} W(x)\, dx = 2 \sum_{n=1}^{\infty} \frac{(-2n)^{2n-1}}{(2n+1)!\mathrm{e}^{2n+1}}.$$

LEMMA 10.3

$$\int_0^{e^{-1}} {}^{\infty}(e^x)\,dx = \sum_{n=0}^{\infty} \frac{(n+1)^{n-2}}{n!e^{n+1}}.$$

Proof

$$\int {}^{\infty}(e^x)\,dx = \sum_{n=0}^{\infty} \frac{(n+1)^n}{(n+1)!} \int x^n dx = \sum_{n=0}^{\infty} \frac{(n+1)^{n-2}}{n!} x^{n+1},$$

whenever $|x| \leq e^{-1}$ using Corollary 9.3, and the Lemma follows from evaluating the integral at $x = e^{-1}$ and at $x = 0$. ∎

Using Lemma 3.3 and W's Definition (2.2), we get,

COROLLARY 10.4

$$\int_0^{e^{-1}} e^{-W(-x)} dx = \sum_{n=0}^{\infty} \frac{(n+1)^{n-2}}{n!e^{n+1}}.$$

Evaluating the integral in the proof of Lemma 10.3 at $x = e^{-1}$ and at $x = -e^{-1}$ instead, we get:

COROLLARY 10.5

$$\int_{-e^{-1}}^{e^{-1}} e^{-W(-x)} dx = 2 \sum_{n=0}^{\infty} \frac{(2n+1)^{2n-2}}{(2n)!e^{2n+1}}.$$

The corresponding results for ${}^{\infty}x$ are slightly more involved.

LEMMA 10.6

$$\int_{e^{-e^{-1}}}^{e^{e^{-1}}} {}^{\infty}x dx = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)^{n-1}}{n!} \int_{-e^{-1}}^{e^{-1}} e^{-t} t^n dt$$

Proof

$$\int {}^{\infty}x dx = \sum_{n=0}^{\infty} \frac{(n+1)^n}{(n+1)!} \int \ln(x)^n dx,$$

whenever $e^{-e^{-1}} \leq x \leq e^{e^{-1}}$ using Corollary 9.3, and

$$\int \ln(x)^n dx = (-1)^n x n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \ln(x)^k.$$

The last sum equals

$$\frac{\Gamma(n+1,\,-\ln(x))}{xn!},$$

therefore

$$\int {}^{\infty}x dx = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)^n}{(n+1)!} \Gamma(n+1,\,-\ln(x)).$$

The Lemma follows from evaluating the integral at $x = \mathrm{e}^{-\mathrm{e}^{-1}}$ and at $x = \mathrm{e}^{\mathrm{e}^{-1}}$, and writing $\Gamma(n+1,\,-\mathrm{e}^{-1}) - \Gamma(n+1, \mathrm{e}^{-1}) = \int_{-\mathrm{e}^{-1}}^{\mathrm{e}^{-1}} \mathrm{e}^{-t} t^n dt$. ■

Using Lemma 3.3 and W's definition (2.2), we get:

Corollary 10.7

$$\int_{\mathrm{e}^{-\mathrm{e}^{-1}}}^{\mathrm{e}^{\mathrm{e}^{-1}}} \mathrm{e}^{-W(-\ln(x))} dx = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)^{n-1}}{n!} \int_{-\mathrm{e}^{-1}}^{\mathrm{e}^{-1}} \mathrm{e}^{-t} t^n dt.$$

Using Corollaries 9.4 and 9.5 we get for the finite cases,

Corollary 10.8 *For* $m \in \mathbb{N}$, *and* $x \in \mathbb{R}$,

$$\int {}^{m}(\mathrm{e}^x)\, dx = \sum_{n=0}^{m} \frac{(n+1)^{n-2}}{n!} x^{n+1} + \sum_{n=m+1}^{\infty} \frac{a_{m,n}}{n+1} x^{n+1}$$

where $a_{m,n}$ *are given by* (8.1).

Corollary 10.9 *For* $m \in \mathbb{N}$, $x > 0$, *and* $b_n = \Gamma(n+1,\,-\ln(x))$,

$$\int {}^{m}x dx = \sum_{n=0}^{m} \frac{(-1)^n (n+1)^{n-1}}{n!} b_n + \sum_{n=m+1}^{\infty} (-1)^n a_{m,n} b_n$$

where $a_{m,n}$ *are given by* (8.1).

Corless *et al*. in [17, p. 2] give the following for the nth derivative of W:

Lemma 10.10

$$\frac{d^n W(x)}{dx^n} = \frac{\mathrm{e}^{-nW(x)} p_n W(x)}{(1+W(x))^{2n-1}}$$

with p_n *satisfying*, $p_1 = 1$ *and* $p_{n+1}(w) = -(nw + 3n - 1)p_n(w) + (1+w)p_n'(w)$.

We finish with one example from differentiation which follows from Lemmas 2.16 and 10.10. Many additional identities can be constructed in a similar spirit.

Corollary 10.11

$$\sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{(n-1)!\mathrm{e}^{n-1}} = \frac{eW(\mathrm{e}^{-1})}{1+W(\mathrm{e}^{-1})}.$$

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