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Walsh–Fourier Series and Their Generalizations in Orlicz Spaces

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For any separable Orlicz space L_N we describe all Orlicz subspaces in which every function can be represented in the norm of L_N by the Vilenkin–Fourier series. For Walsh–Fourier partial sums $S_n^w f$ a more precise result is obtained.

INTRODUCTION

Every Vilenkin system forms a Schauder basis in L^p $(1 < p < \infty)$ [26, 27, 37], from which it follows by means of interpolation the same statement for all reflexive Orlicz spaces (see [25]). In [10] it was proved that these systems are not bases in nonreflexive Orlicz spaces. In this paper we are concerned with all separable Orlicz spaces. Let us denote by L_N such a space. We only consider bounded Vilenkin systems. We describe the maximal Orlicz subspace of L_N in which every function can be represented by Vilenkin–Fourier series in the norm of L_N. This maximal subspace is the space L_{R_N} (see (10) for the definition). A particular Vilenkin system is the Walsh system. In this case, we improve our

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previous result; in fact the set $\{f \in L^1;\ \forall g$ measurable, $|g| = |f|$, $\lim_{n\to\infty} \|g - S_n^w g\|_N = 0\}$ is just L_{R_N}. For L^1 this statement is proved in [9]. Our proof is essentially based on Theorem 3.4 due to Burkholder and Gundy (see [7, 5, 6]). The scheme of the proof of Theorem 3.1 was first applied in [9] (see also [8]) for the space L^1. We remark that the problem of representation of functions by the classical systems in integral metric was due to P. L. Ul'yanov [31]. Among the other works connected with our investigation we mention only [32, 18–20, 8, 29].

1. PRELIMINARIES

Let $m = \{m_i\}_{i=0}^{\infty}$ be a bounded sequence of integers, $2 \le m_i \le C$, $i \in \mathbb{N}$. Define the group G_m as the set of all sequences $x = (x_0, x_1, \dots)$ $(0 \le x_k < m_k, x_k \in \mathbb{N}, k \in \mathbb{N})$ with the group-operation $x + y = ((x_0 + y_0) \bmod m_0, (x_1 + y_1) \bmod m_1, \dots)$ $(x, y \in G_m)$.

The topology of G_m is given by the neighbourhoods

$$I_n(x) = \{y \in G_m : y_0 = x_0, \dots, y_{n-1} = x_{n-1}\} \qquad (x \in G_m, n \in \mathbb{N}) \quad (1)$$

thus G_m forms a compact abelian group. Let us introduce in G_m the normalized Haar measure μ. If $M_0 = 1$, $M_{k+1} = m_k M_k$, $k \in \mathbb{N}$ then the group G_m can be transformed in the interval $[0,1]$ by means of the mapping

$$G_m \ni x \mapsto \sum_{j=0}^{\infty} \frac{x_j}{M_{j+1}} \in [0,1].$$

If we disregard the countable set of m_i rationals, this mapping is one-to-one, onto, and measure preserving.

For each $k \in \mathbb{N}$, $x \in G_m$ we define $r_k(x) = \exp(2\pi i x_k / m_k)$. If $n \in \mathbb{N}$ then there exists a unique representation

$$n = \sum_{k=0}^{\infty} n_k M_k, \qquad 0 \le n_k < m_k, \quad n_k \in \mathbb{N}, k \in \mathbb{N}.$$

Let $\phi_n = \prod_{k=0}^{\infty} r_k^{n_k}$. The functions ϕ_n are the characters of G_m and they form a complete orthonormal system on G_m, which is called the generalized Walsh system or Vilenkin system, generated by the sequence m. For the case $m_i = 2$, $i = 0, 1, \dots, G_m$ is the dyadic group, the r_n are the Rademacher functions, and ϕ_n the Walsh functions w_n.

We consider Fourier series with respect to the system $\{\phi_n\}$.

Let $D_n = \sum_{i=0}^{n-1} \phi_i$, $n = 1, 2, \dots,$ be the Dirichlet kernel. Then we have the following representation of the Dirichlet kernel (see [27]):

$$D_n = \phi_n \sum_{k=0}^{\infty} \left(\overline{\sum_{j=1}^{n_k} (r_k)^j} \right) D_{M_k}.$$

For $f \in L^1(G_m)$

$$S_n^{\phi}(f, x) = \int_{G_m} f(t) D_n(x \dot{-} t)\, d\mu(t) \tag{2}$$

denotes the nth partial Fourier sum of f with respect to the system $\{\phi_n\}$.

THEOREM 1.1 [26, 27, 37]. *There are constants C and C_p such that for* $n = 1, 2, \dots$

(a) $\|S_n^{\phi} f\|_{L^p(G_m)} \le C_p \|f\|_{L^p(G_m)}$ $\forall f \in L^p(G_m)$, $1 < p < \infty$.

(b) $\mu\{x \in G_m : |S_n^{\phi}(f, x)| > y\} \le C y^{-1} \|f\|_{L^1(G_m)}$ $\forall f \in L^1(G_m)$ $\forall y > 0$.

LEMMA 1.1 [10]. *There are constants C_1, C_2 and a sequence $\{V_p\}$ such that*

$$1 < \frac{V_{p+1}}{V_p} < C_1 \tag{3}$$

and if

$$M_q \le V_p < M_{q+1}, \qquad V_p = \sum_{k=0}^{q} n_k M_k, \tag{4}$$

then

$$\mu\left\{x \in G_m : \left|D_{V_p}(x)\right| > t\right\} \ge \frac{C_2}{t} \qquad \left(1 \le t \le \frac{V_p}{\pi}\right), \tag{5}$$

where

$$D_{V_p} = \phi_{V_p} \sum_{k=0}^{q} \left(\overline{\sum_{j=0}^{n_k} r_k^j} \right) D_{M_k} \tag{6}$$

$$D_{M_k} = M_k \mathbf{1}_{I_k} \tag{7}$$

and $I_k = I_k(0)$ is defined as in (1).

Let N be a function generating the Orlicz space $L_N = L_N(G_m)$, i.e., $N\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a convex continuous function

$$\lim_{x \to \infty} \frac{N(x)}{x} = \lim_{x \to 0} \frac{x}{N(x)} = \infty. \tag{8}$$

The space L_N endowed with the norm

$$\|f\|_N = \inf\left\{k > 0: \int_{G_m} N\left(\frac{|f(x)|}{k}\right) dx \le 1\right\} \tag{9}$$

is a Banach space.

The complementary Young function to N is given by

$$N^*(u) = \sup_{v \ge 0} (uv - N(v)).$$

Let R_N be a function generating the Orlicz space L_{R_N} such that

$$R_N(u) = u \int_1^u t^{-2} N(t)\, dt \qquad (u \ge 2). \tag{10}$$

In what follows the definition of R_N for $0 \le u < 2$ does not play any role.

The following properties are equivalent:

$$\text{the function } N \text{ satisfies } \Delta_2 \text{ condition, i.e.,} \tag{11}$$

$$\exists C > 0\ \exists u_0 \ge 0\ \forall u \ge u_0, \qquad N(2u) \le CN(u),$$

$$\text{the Orlicz space } L_N \text{ is separable,} \tag{12}$$

$$\|f\|_N < \infty \Leftrightarrow \int_{G_m} N(|f|) < \infty, \tag{13}$$

$$\exists r \ge 1\ \exists u_0 \ge 1\ \exists C > 0\ \forall u \ge u_0, \qquad u^r \int_u^\infty t^{-r-1} N(t)\, dt \le CN(u). \tag{14}$$

For (11) ⇔ (12) ⇔ (13) see [14]; for (11) ⇔ (14) see [30].

We note that the Δ_2 condition for N implies the properties

$$\text{the function } R_N \text{ satisfies } \Delta_2 \text{ condition,} \tag{15}$$

$$\exists C > 1\ \exists u_0 \ge 0\ \forall u \ge u_0, \qquad N(u) \le CR_N(u), \tag{16}$$

or the equivalent form

$$L_{R_N} \subset L_N. \tag{17}$$

$$L_N \text{ is reflexive if and only if } N \text{ and } N^* \text{ both satisfy the } \Delta_2 \text{ condition.} \tag{18}$$

We will use the following variant of the Marcinkiewicz interpolation theorem (see [38; 30, p. 151 and (14)]).

MARCINKIEWICZ INTERPOLATION THEOREM (see [38; 30, p. 151]). *Let A be a quasilinear operator $A: L^1 \to L^0$, which is of type (p, p), $1 < p < \infty$, and weak type $(1, 1)$. Let $N: \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function satisfying the Δ_2 condition.*

Then for f such that $\int_{G_m} R_N(|f|) < \infty$ we have

$$\int_{G_m} N(|Af|) \leq C\left(1 + \int_{G_m} R_N(|f|)\right).$$

(Here L^0 denotes the space of μ measurable functions on G_m.)

LEMMA 1.2. *Let N satisfy the Δ_2 condition. Then $L_N = L_{R_N}$ if and only if L_N is reflexive.*

Proof. The proof follows from the chain of equivalences:

$$\left[L_N = L_{R_N}\right] \Leftrightarrow (17)\left[L_N \subset L_{R_N}\right] \Leftrightarrow [14]\left[N^* \text{ satisfies the } \Delta_2 \text{ condition}\right]$$
$$\Leftrightarrow (18)\left[L_N \text{ is reflexive}\right].$$

2. VILENKIN–FOURIER SERIES IN SEPARABLE ORLICZ SPACES

THEOREM 2.1. *Let L_N and L_Q be Orlicz spaces and L_N be separable. Then the following assertions are equivalent*

(1) $L_Q \subset L_{R_N}$;
(2) $\sup_n \|S_n^\phi\|_{L_Q \to L_N} < \infty$;
(3) *for every $f \in L_Q$ we have*

$$\left\|f - S_n^\phi f\right\|_N \to 0 \qquad (n \to \infty).$$

Proof. The scheme of the proof is [(2) ⇔ (1)], [(1), (2) ⇒ (3) ⇒ (1)].

(1) ⇒ (2). Let $f \in L_Q$. Then $f \in L_{R_N}$ and (by (13)) $\int R_N(|f|) < \infty$. By virtue of the Marcinkiewicz interpolation theorem and Theorem 1.1 for all $n = 1, 2, \ldots$ we have

$$\int N\left(\left|S_n^\phi f\right|\right) \leq C\left(1 + \int R_N(|f|)\right) \leq C\left(1 + \int Q(|f|)\right).$$

This implies (2).

(2) ⇒ (1). By virtue of the convexity of Q (see also (8)) and the increase of M_n there exists an integer

$$q_0 = \min\{q : Q^{-1}(M_q) < M_q\}. \tag{19}$$

We put

$$p_0 = \min\{p : V_p > M_{q_0}\} \tag{20}$$

and for $p \geq p_0$, we consider V_p defined as in Lemma 1.1.

We put

$$f_{V_p}(x) = M_{q+1}\mathbf{1}_{I_{q+1}}(x)\,\phi_{V_p}(x), \qquad x \in G_m. \tag{21}$$

Then (see (2), (6), (1), (7)), we have

$$\left|S^{\phi}_{V_p}(f_{V_p})\right| = \left|D_{V_p}\right|. \tag{22}$$

The boundedness of the sequence (m_i) implies the estimate (see also (9))

$$\|f_{V_p}\|_Q = \frac{M_{q+1}}{Q^{-1}(M_{q+1})} \leq C\,\frac{M_q}{Q^{-1}(M_q)}.$$

If (2) holds then (see (12), (5), (10), (11)) for $p \geq p_0$ we obtain (see also (19), (22))

$$\begin{aligned}
1 &\geq \int_{G_m} N\left(\frac{1}{C\|f_{V_p}\|_Q}\left|S^{\phi}_{V_p}(f_{V_p})\right|\right) \geq C\int_{G_m} N\left(\frac{Q^{-1}(M_q)}{M_q}\left|D_{V_p}\right|\right) \\
&\geq C\int_{Q^{-1}(M_q)/M_q}^{V_pQ^{-1}(M_q)/M_q} N'(t)\,\mu\left\{x \in G_m : \left|D_{V_p}(x)\right| > t\,\frac{M_q}{Q^{-1}(M_q)}\right\} dt \\
&\geq C\,\frac{Q^{-1}(M_q)}{M_q}\int_{Q^{-1}(M_q)/M_q}^{V_pQ^{-1}(M_q)/M_q} t^{-2}N(t)\,dt \\
&\geq C\,\frac{Q^{-1}(M_q)}{M_q}\int_{1}^{V_pQ^{-1}(M_q)/\pi M_q} t^{-2}N(t)\,dt.
\end{aligned}$$

In the last inequality we used the convexity of Q (see also (4)). For $t \geq V_{p_0}$ using the estimates (3) and (15) we have

$$R_N(Q^{-1}(t)) \leq Ct$$

that implies (1) (see (16), (17)).

((1), (2)) ⇒ 3. From part (1) and (17) it follows that for every polynomial T_n of order n with respect to the system $\{\phi_n\}$ we have (see also [38])

$$\|f - S_n^{\phi} f\|_N \leq \|f - T_n\|_N + \|S_n^{\phi}\|_{L_{R_N} \to L_N} \|f - T_n\|_N$$
$$\leq \left(1 + \|S_n^{\phi}\|_{L_q \to L_N}\right) \|f - T_n\|_N .$$

The density of the set of polynomials T_n in the separable Orlicz space L_N implies (3).

(3) ⇒ (2). This implication is trivial.

Another formulation for the previous results may be useful.

THEOREM 2.2. *Let N and Q be increasing nonnegative functions on $\mathbb{R}^+$ with N satisfying the Δ_2 condition and Q convex.*

Then the following assertions are equivalent:

(a) $R_N(t) \leq CQ(t)$, $t \geq t_0 > 0$

(b) $\int_{G_m} N(|S_n^{\phi} f|) \leq C(1 + \int_{G_m} Q(|f|))$.

COROLLARY 2.1. *Let $N: \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function.*

Then the following assertions hold:

(a) $[\exists C > 0 \quad \forall f \in L^1 \quad \forall n \in \mathbb{N}, \quad \int_{G_m} N(|S_n^{\phi} f|) \leq C(1 + \|f\|_{L^1})] \Leftrightarrow [\int_1^\infty t^{-2} N(t)\, dt < \infty]$;

(b) $[\exists C > 0, \quad \forall n \in N, \quad \int_{G_m} N(|S_n^{\phi} f|) \leq C(1 + \int |f| \ln \ln(e + |f|))] \Leftrightarrow [\exists C > 0\ \exists u_0 > 1\ \forall u \geq u_0,\ N(u) \leq C(u/\ln u)]$;

(c) $[\exists C > 0 \quad \forall n \in \mathbb{N}, \quad \int_{G_m} N(|S_n^{\phi}(f)|) \leq C(1 + \int |f| \ln^{\alpha+1}(1 + |f|)), \alpha > 0] \Leftrightarrow [\exists C > 0\ \exists u_0 > 1\ \forall u \geq u_0,\ N(u) \leq Cu \ln^{\alpha} u]$.

Remark. Corollary 2.1 presents the analogue of the corresponding results for the trigonometric case [15, 16, 38].

3. WALSH–FOURIER SERIES IN NONREFLEXIVE SEPARABLE ORLICZ SPACES

For Walsh–Fourier series Theorem 2.1 may be improved. In this part $\{w_n\}$ denotes the Walsh system and $S_n^w f$ the nth partial Walsh–Fourier sum of $f \in L^1[0,1]$.

The main result of this part is

THEOREM 3.1. *Let L_N be a separable nonreflexive Orlicz space. Then for every $f \in L_N \setminus L_{R_N}$ there exists a measurable function g on* $[0,1]$ *such that $|f| = |g|$ and*

$$\sup_n \|S_n^w g\|_N = \infty.$$

The proof is based on a few propositions.

PROPOSITION 3.1 [17; 22; 13, p. 39]. *Let* $\{f_n\}_{n=0}^{\infty}$ *be in* $L^0[0,1]$. *The following assertions are equivalent*:

(a) $\sum_{n=0}^{\infty} r_n(t) f_n(x)$ *converges for almost every* x *and* $t \in [0,1]$.

(b) $\sum_{n=0}^{\infty} r_n(t) f_n(x)$ *converges in measure* (*with respect to* x) *for almost every* $t \in [0,1]$.

(c) $\sum_{n=0}^{\infty} f_n^2(x)$ *is finite for almost every* $x \in [0,1]$.

PROPOSITION 3.2 [23; 13, p. 30]. *There exists a constant* $C > 0$ *such that for every* m *and every sequence of scalars* $\{\mathscr{C}_n\}$ *we have*

$$\mu\left(\left\{t \in [0,1] : \left|\sum_{n=0}^{m} r_n(t)\mathscr{C}_n\right| > \frac{1}{2}\left(\sum_{n=0}^{m} \mathscr{C}_n^2\right)^{1/2}\right\}\right) \geq C. \tag{23}$$

PROPOSITION 3.3 [24]. *Let* $N: \mathbb{R}^+ \to \mathbb{R}^+$ *be an increasing function which satisfies the* Δ_2 *condition. If* $\sum_{n=0}^{\infty} \mathscr{C}_n^2$ *is finite then* $\int N(|\sum_{n=0}^{\infty} r_n(t)\mathscr{C}_n|)\,dt$ *is finite*.

For $f \in L^1$ we denote by

$$Pf = \left\{\|f\|_{L^1}^2 + \sum_{k=0}^{\infty} \left[S_{2^{k+1}}^w f - S_{2^k}^w f\right]^2\right\}^{1/2}$$

the Paley function and by

$$S^*(f,x) = \sup_n \left|S_{2^n}^w(f,x)\right|$$

the majorants of the Walsh–Fourier partial sums of order 2^n.

We will use a quite particular case of the Burkholder–Gundy theorem [7] (see also [5, 6]).

THEOREM 3.4 [7]. *Let* L_N *be a separable Orlicz space. Then* S^*f *is in* L_N *if and only if* Pf *is in* L_N.

The following statement is well known for $N(u) = u^p$, $1 \leq p < \infty$ [17]; see also [13]. We use the approach of [13, p. 41].

PROPOSITION 3.5. *Let* L_N *be a separable Orlicz space and* $f \in L_N$ *such that for every* $t \in [0,1]$, $\|\sum_{n=0}^{\infty} r_n(t) f_n\|_N$ *is finite. Then* $\|(\sum_{n=0}^{\infty} f_n^2)^{1/2}\|_N$ *is finite*.

Proof. We have that for every $t \in [0,1]$, see (13),

$$\int N\left(\left|\sum_{n=0}^{\infty} r_n(t) f_n(x)\right|\right) dx = \eta(t) < \infty.$$

Thus $\sum r_n(t) f_n(x)$ converges in measure (with respect to x) and by Proposition 3.1 it converges for almost every (t,x) in $[0,1] \times [0,1]$. By Proposition 3.2 there exists a constant C satisfying (23). Take $\delta = C/2$. There exist a measurable subset E of $[0,1]$ and a constant $K \geq 0$ such that $m(E) > 1 - \delta$ and

$$\forall t \in E, \qquad \eta(t) \leq K.$$

Then by using Fubini's theorem we get that

$$\int \left[\int_E N\left(\left|\sum_{n=0}^{\infty} r_n(t) f_n(x)\right|\right) dt \right] dx = \int_E \eta(t)\, dt \leq K\mu(E) \leq K. \quad (24)$$

By Proposition 3.1, we have that $\sum_{n=0}^{\infty} f_n^2(x)$ is finite for almost every $x \in [0,1]$. Take such an x and put

$$Q_m = \left\{ t \in [0,1] : \left|\sum_{n=0}^{m} r_n(t) f_n(x)\right| \geq \frac{1}{2}\left(\sum_{n=0}^{m} f_n^2(x)\right)^{1/2} \right\}.$$

By Proposition 3.2, $\mu(Q_m) \geq C$ and $\mu(E \cap Q_m) \geq C/2$.

By Proposition 3.3, we also have $\int_0^1 N(|\sum_{n=0}^{\infty} r_n(t) f_n(x)|)\, dt$ is finite and

$$\begin{aligned}
\int_0^1 N\left(\left|\sum_{n=0}^{m} r_n(t) f_n(x)\right|\right) dt &\geq \int_{E \cap Q_m} N\left(\left|\sum_{n=0}^{m} r_n(t) f_n(x)\right|\right) dt \\
&\geq \frac{C}{2} N\left(\frac{1}{2}\left(\sum_{n=0}^{m} f_n^2(x)\right)^{1/2}\right) \\
&\geq \frac{C}{2} N\left(\left(\sum_{n=0}^{m} f_n^2(x)\right)^{1/2}\right). \quad (25)
\end{aligned}$$

Combining (24) and (25) we get the result.

Proof of Theorem 3.1. We first remark that by virtue of Lemma 1.2

$$L_N \setminus L_{R_N} \neq \emptyset.$$

The condition of Theorem 3.1 and (13), (12), (15) imply that

$$\int N(f) < \infty \qquad \text{and} \qquad \int R_N(f) = \infty. \quad (26)$$

Without loss of generality we can suppose that f is positive.

We suppose now that

$$\sup_n \| S_n^w f \|_N < \infty \tag{27}$$

(otherwise Theorem 3.1 is proved).

We first show that S^*f is not in L_N. Using the estimate (see [9, p. 73])

$$\frac{1}{t} \int_{\{g>t\}} g \leq 5\mu(\{x: S^*(g,x) > t\}) \qquad \forall g \in L^1 \ \forall t > \|g\|_{L^1},$$

we obtain for $f > 0$, see (26),

$$\begin{aligned}
\infty &= \int_0^1 R_N(f) = \int_{\{0 \leq f < 2\}} R_N(f) + \int_{\{f \geq 2\}} R_N(f) \\
&\leq C + \int_0^1 f(x) \int_1^{f(x)} t^{-2} N(t)\, dt\, dx \\
&\leq C + \left(\int_1^{\max(1, \|f\|_{L^1})} + \int_{\max(1, \|f\|_{L^1})}^{\infty} \right) \frac{N(t)}{t^2} \int_{\{f>t\}} f \\
&\leq C + 5 \int_1^{\infty} \frac{N(t)}{t} \mu(\{x: S^*(f,x) > t\})\, dt \\
&\leq C + 5 \int_1^{\infty} N'(t) \mu(\{x: S^*(f,x) > t\})\, dt = C + 5 \int N(S^*f).
\end{aligned}$$

By Theorem 3.4 of Burkholder and Gundy it implies $\int N(Pf) = \infty$ and (see (13)) $\|Pf\|_N = \infty$.

Then by Proposition 3.5, there exists $\epsilon_n = \pm 1$ such that

$$\left\| \sum_{n=0}^{\infty} \epsilon_n (S_{2^{n+1}}^w f - S_{2^n}^w f) \right\|_N = \infty,$$

hence there exists $\beta_n \in \{0, 1\}$ such that

$$\left\| \sum_{n=0}^{\infty} \beta_n (S_{2^{n+1}}^w f - S_{2^n}^w f) \right\|_N = \infty.$$

Put

$$N_k = \sum_{n=0}^{k} \beta_n 2^n.$$

It is well-known [13, p. 142] that for $g \in L^1$,

$$w_{N_k} S^w_{N_k}(g w_{N_k}) = \sum_{n=0}^{k} \beta_n (S^w_{2^{n+1}} g - S^w_{2^n} g).$$

Thus

$$\infty = \lim_k \left\| S^w_{N_k}(f w_{N_k}) \right\|_N = \lim_k \left\| S^w_{N_k}\big(f \mathbf{1}_{w_{N_k} = +1}\big) - S^w_{N_k}\big(f \mathbf{1}_{w_{N_k} = -1}\big) \right\|_N .$$

This shows that there exist measurable subsets E_k of $[0, 1]$ such that

$$\sup_k \left\| S^w_{N_k}(f \mathbf{1}_{E_k}) \right\|_N = \infty. \tag{28}$$

We suppose that for every measurable subset E of $[0, 1]$,

$$\sup_n \| S^w_n(\mathbf{1}_E f) \|_N < \infty. \tag{29}$$

We consider the complete metric space $\mathscr{E}$ of all measurable subsets of $[0, 1]$ endowed with the metric d

$$d(E_1, E_2) = \| \mathbf{1}_{E_1} - \mathbf{1}_{E_2} \|_{L^1} \qquad \forall E_1, E_2 \in \mathscr{E}$$

and we represent

$$\mathscr{E} = \bigcup_{m=1}^{\infty} F_m ,$$

where $F_m = \{E \in \mathscr{E} : \sup_n \|S^w_n(\mathbf{1}_E f)\|_N \le m\}$ is closed.

Then by Baire's theorem there exists m_0 such that F_{m_0} contains a ball $B(E_0, r_0)$.

It follows that if $\mu(E) \le r_0$ then

$$\sup_n \| S^w_n(\mathbf{1}_E f) \|_N \le \sup_n \left\| S^w_n\big(\mathbf{1}_{E \cup E_0} f\big) \right\|_N + \sup_n \left\| S^w_n\big(\mathbf{1}_{E_0 \setminus E} f\big) \right\|_N \le 2 m_0 .$$

For each E of $\mathscr{E}$ we can write $\mathbf{1}_E f = \sum_{j=1}^{l} \mathbf{1}_{E_j} f$ with $l \le r_0^{-1}$ and $m(E_j) \le r_0$. We then obtain

$$\sup_n \| S^w_n(\mathbf{1}_E f) \|_N \le 2 m_0 r_0^{-1}.$$

This gives a contradiction with (28). Thus there exists $E \in \mathscr{E}$ such that

$$\sup_n \| S^w_n(\mathbf{1}_E f) \|_N = \infty.$$

Then we can define the function g as (see also (27))

$$g(t) = \left(1 - 2\mathbf{1}_E(t)\right)f(t),$$

and Theorem 3.1 is proved.

4. REMARKS

4.1. In the reflexive case Theorem 2.1 gives the well-known results mentioned above [10]; see Lemma 1.2.

4.2. It follows from Theorem 2.1 that if N satisfies the Δ_2 condition then the condition $[\exists C > 0\ \ \exists u_0 > 1\ \ \forall u \geq u_0,\ \ Q(u) \geq CN(u)\ln u] \Rightarrow [\forall f \in L_Q, \|f - S_n^{\phi} f\|_N \to 0,\ u \to \infty]$.

4.3. Using the approach of [28] we can obtain results in dual spaces. For example, it is true that:

THEOREM 4.1. *Let L_Q be an Orlicz space. Then the following assertions are equivalent*

(a) $L_Q \supset L_{e^u}$;

(b) *for every continuous function* f, $\|f - S_n^{\phi} f\|_Q \to 0,\ n \to \infty$.

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