1) Stern-Gerlach spin measurement & Bloch sphere
2) B-field

Lecture

Crommie #3

So far we have derived the origin of spin-½ states and the spin operators that act on such states:

$$\vec{S} = S_x \hat{x} + S_y \hat{y} + S_z \hat{z}, \quad S^2 = S_x^2 + S_y^2 + S_z^2$$

$$S^2 = \tfrac{3}{4}\hbar^2 \sigma_0, \quad S_x = \tfrac{\hbar}{2}\sigma_1, \quad S_y = \tfrac{\hbar}{2}\sigma_2, \quad S_z = \tfrac{\hbar}{2}\sigma_3, \quad \sigma_0, \sigma_1, \sigma_2, \sigma_3 = \text{Pauli Spin Matrices}$$

Spin-½ eig. states = $|0\rangle, |1\rangle$

These are simult. eig. states of S^2 & S_z:

$S^2|0\rangle = \tfrac{3}{4}\hbar^2|0\rangle$

$S^2|1\rangle = \tfrac{3}{4}\hbar|1\rangle$

$S_z|0\rangle = \tfrac{\hbar}{2}|0\rangle \rightarrow$ spin up ↑

$S_z|1\rangle = -\tfrac{\hbar}{2}|1\rangle \rightarrow$ spin down ↓

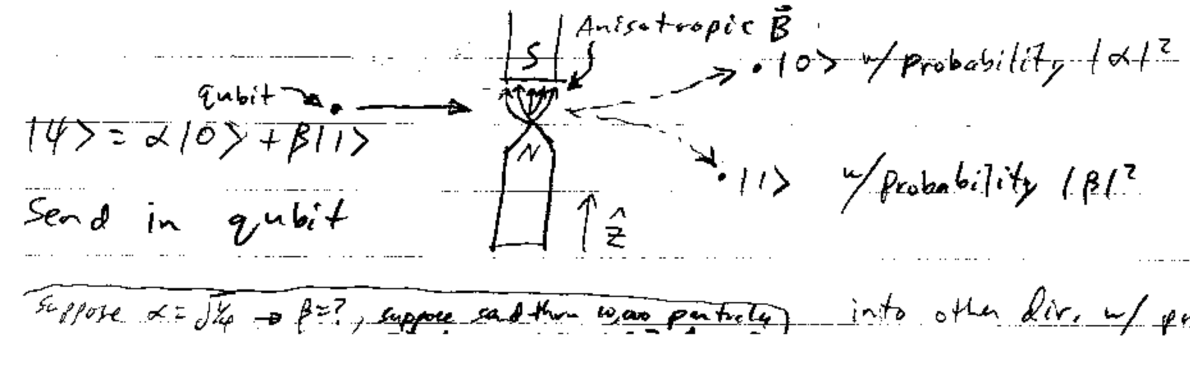
Most General Spin-½ state $\Rightarrow |\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix}\alpha \\ \beta\end{pmatrix}$ ← Clearly a good qubit

So, now we have a good qubit system, but how do we actually manipulate or measure the state of a spin qubit?

One possible answer: Stern-Gerlach Device (S-G)

S-G takes advantage of the fact that spin leads to a magnetic moment, $\vec{\mu}$. The magnetic moment gives us a way to experimentally "grab" the spin via the interaction $\hat{H} = -\vec{\mu}\cdot\vec{B}$

What is a S-G Device? It is a funny magnet:



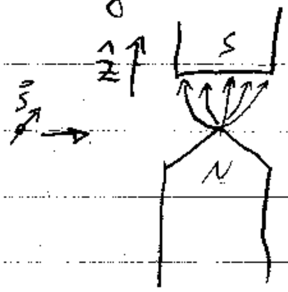
S-G performs "measurement" & spits qubit into one direction w/ prob. $|\alpha|^2$ [for $|0\rangle$] and into other dir. w/ prob. $|\beta|^2$ [for $|1\rangle$]

Suppose $\alpha = \sqrt{1/4} \rightarrow \beta = ?$, suppose send thru 10,000 particles

Measures Comp. of $\vec{S} \parallel \hat{z}$!

How Does S-G device work? "Plausability argument"

Think "semi-classically", consider classical Force acting on mag. moment $\vec{\mu}$ of spin sent through center of S-G:



$$E = -\vec{\mu}\cdot\vec{B} \Rightarrow \vec{F}_{spin} = -\vec{\nabla}E = \vec{\nabla}(\vec{\mu}\cdot\vec{B})$$

Consider spin sent through center of S-G device

$$\Rightarrow \vec{B} = B(z)\hat{z} \quad \Rightarrow \quad \vec{F}_{\substack{spin \\ at\ center}} = \vec{\nabla}(\mu_z B(z)) = \mu_z \frac{\partial B}{\partial z}\hat{z}$$

So, spin gets a "push" in z-direction: How hard is push?

$$F_z = \mu_z \frac{\partial B}{\partial z} = \frac{-e}{m} S_z \frac{\partial B}{\partial z} = +\frac{e}{m}\left|\frac{\partial B}{\partial z}\right| S_z \quad \left(\text{since } \frac{\partial B}{\partial z} < 0\right)$$

$[\vec{\mu} = \frac{e}{m}\vec{S}]$

$\Rightarrow$ What are possible values of S_z? For spin-½ $\Rightarrow S_z = \pm\frac{\hbar}{2}$

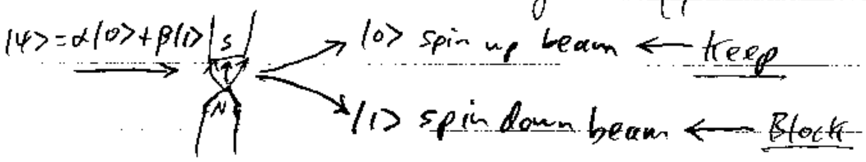
$\therefore$ spin-up qubits ($S_z = +\frac{\hbar}{2}$) get pushed "up" and spin-down qubits ($S_z = -\frac{\hbar}{2}$) get pushed down. $\Rightarrow$ Get real spatial separation of qubit eigenstates. $\rightarrow$ A spin measurement!

[First done in 1922 by Stern-Gerlach for Ag atoms]

Qubit "initialization"

Application: Prepare a beam of spin qubits in pure $|\psi\rangle = |0\rangle$ state from a beam of spins in a randomly prepared state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

How to do it: Send spins through a S-G device aligned $\parallel \hat{z}$ and block the "$|1\rangle$" beam:



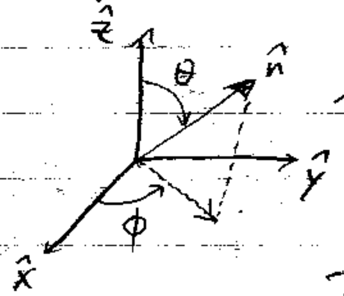
← Easy!

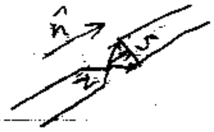
New Question: Suppose I have a beam of spins in the pure state $|\psi\rangle = |0\rangle$. Is it possible to create a new beam of spins in the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$?
(mod global phase factor)

Answer: Yes! You can use a S-G device.

How?: [a little bit of work!]

Consider a S-G device oriented parallel to the unit vector $\hat{n}$, which points in the (θ, ϕ) direction in spherical coordinates:





what happens if spin qubit in $|\psi\rangle = |0\rangle$ state is passed through this?

This is like a rotation of the coordinate system.

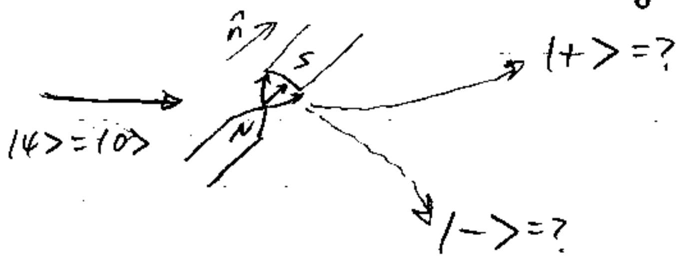
Remember: Qubit feels force $\vec{F} = \vec{\nabla}(\vec{\mu}\cdot\vec{B})$

But now $\vec{B} \parallel \hat{n}$ at center of S-G $\Rightarrow$ $\vec{F} = F_n \hat{n} = \frac{e}{m}\left|\frac{\partial B}{\partial z_n}\right| S_n \hat{n}$

So, measure comp. of $\vec{S} \parallel \hat{n}$!! where $S_n = \vec{S}\cdot\hat{n}$ = component of $\vec{S}$ in dir. $\hat{n}$

Spins $\therefore$ get pushed in the $\pm\hat{n}$ direction depending on whether they are "up" or "down" eigenstates of "S_n".

By sending the beam of qubits through the rotated S-G, we separate it into 2 new beams, $|+\rangle$, $|-\rangle$, that are each in an eigenstate of $\hat{S}_n$:



So, we start w/ eigenstate of S_z, but we end up w/ eig state of "S_n".

$\Rightarrow$ WHAT ARE eig. states of S_n??

$\star$

To answer this question we must Find the $|+\rangle, |-\rangle$ eig. states of $\hat{S}_n$ in the $|0\rangle, |1\rangle$ basis (since we start w/ $|0\rangle$)

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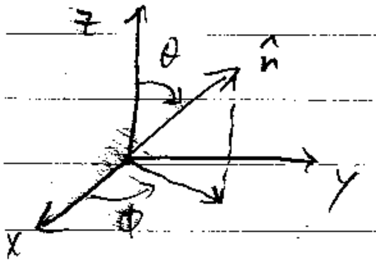
~~Must Find eig. states of S_n to understand what rotated S-G does to beam of spins originally in $|\psi\rangle = |0\rangle$ state.~~

$\Rightarrow$ Find Eigenstates of $S_n = \vec{S} \cdot \hat{n}$: [First guess: How many? why?]

technique: Express S_n as a 2×2 matrix in the $|0\rangle, |1\rangle$ basis. $\Rightarrow$ Diagonalize this matrix to get $|+\rangle, |-\rangle$ eigenstates in terms of $|0\rangle, |1\rangle$

$\Rightarrow$ Can easily see how states transform via the rotated S-G device.

Find S_n : $\hat{n}$ defined in spherical coords, but it's easier if we express it in cartesian coords : [since start w/ eig. states of S_z]



$$\Rightarrow \hat{n} = \underbrace{\sin\theta\cos\phi}_{n_x}\, \hat{x} + \underbrace{\sin\theta\sin\phi}_{n_y}\, \hat{y} + \underbrace{\cos\theta}_{n_z}\, \hat{z}$$

$$\Rightarrow S_n = \vec{S} \cdot \hat{n} = S_x n_x + S_y n_y + S_z n_z$$

$$\Rightarrow \hat{S}_n = \hat{S}_x \sin\theta\cos\phi + \hat{S}_y \sin\theta\sin\phi + \hat{S}_z \cos\theta$$

$\Rightarrow$ Express S_n as a 2×2 matrix using Pauli Matrices:

$$\hat{S}_n = \sin\theta\cos\phi \cdot \frac{\hbar}{2}\hat{\sigma}_1 + \sin\theta\sin\phi \cdot \frac{\hbar}{2}\hat{\sigma}_2 + \cos\theta \cdot \frac{\hbar}{2}\hat{\sigma}_3$$

$\Rightarrow$ Plug in $\sigma_1, \sigma_2, \sigma_3$, add matrices $\Rightarrow$ $S_n = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}$

$\Rightarrow$ To find results of measuring observable "S_n" $\Rightarrow$ Must find eig. states / eig. values of $\hat{S}_n$ $\Rightarrow$ Diagonalize this matrix

$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \lambda \begin{pmatrix} e \\ f \end{pmatrix} \rightarrow$ Find λ, e, f $\Rightarrow \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$, etc.

HOMEWORK

ANSWER: Eig. values are $s_n = +\hbar/2, -\hbar/2$ (makes sense; why?)

Eig. states are $|0\rangle_n = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle \quad (s_n = +\frac{\hbar}{2})$

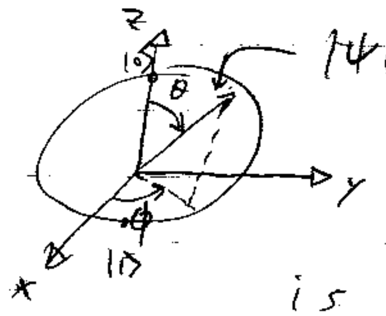
$$|1\rangle_n = -e^{-i\phi}\sin\frac{\theta}{2}|0\rangle + \cos\frac{\theta}{2}|1\rangle \quad (s_n = -\frac{\hbar}{2})$$

(remember, these states are arbitrary up to a multiplicative phase factor)

i.e., $\hat{S}_n|0\rangle_n = +\frac{\hbar}{2}|0\rangle_n, \quad \hat{S}_n|1\rangle_n = -\frac{\hbar}{2}|1\rangle_n$

But wait! $|0\rangle_n$ looks familiar! Where have we seen it before?

It is exactly the qubit that is defined at the (θ, ϕ) pt. on the Bloch Sphere!!



$$|\psi(\theta,\phi)\rangle = \underbrace{\cos\frac{\theta}{2}}_{\alpha}|0\rangle + \underbrace{e^{i\phi}\sin\frac{\theta}{2}}_{\beta}|1\rangle$$ by Def'n of Bloch Sphere.

Now see that the Bloch Sphere is more than just an arbitrary mathematical construct! It gives a direct recipe for initializing Qubits in a laboratory:

Recipe for Constructing Arbitrary Spin state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ in laboratory: (to w/in arb. global phase)

1) Find corresponding angles (θ, ϕ) on Bloch Sphere ($\hat{n}$-dir.):
$\tan\frac{\theta}{2} = \left|\frac{\beta}{\alpha}\right|, \quad e^{i\phi} = \beta/|\beta| \quad$ [since $\beta = |\beta|e^{i\phi}$] ← [H.W.!!]
2) Orient S-G device in real space so that $N \to S$ pts. in (θ,ϕ) direction.
3) Send spins through S-G and capture spin-up beam! → DONE!

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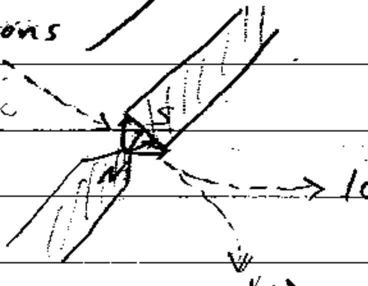
Send electrons into rotated S-G Device $\hat{n} \parallel (\theta, \phi)$, where $\tan\frac{\theta}{2} = \frac{|\beta|}{|\alpha|}$, $e^{i\phi} = \frac{\beta}{|\beta|}$



$$|0\rangle_n = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle = \underline{\alpha|0\rangle + \beta|1\rangle}$$

$|1\rangle_n$

"Spin-up" beam is in exactly the state we want!

∴ We can thus "dial-up" any state on the Bloch sphere by "pointing" to it with a S-G device.

Important Point: This is a useful technique for generating $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ (for beams of random particles) via collapse of the wave-function. It is a statistical process, any given electron sent into S-G has a probability of coming out in $|0\rangle_n$ state (it might collapse into $|1\rangle_n$ state instead!) (H.W. Problem!). This is NOT a UNITARY transformation, $\hat{U}$!! (why?) (not reversible via adjoint)

NEW Question: Suppose we have a single electron in the state $|\psi\rangle = |0\rangle$. How do we transform the state of this one electron into the new state $|\psi'\rangle = \alpha|0\rangle + \beta|1\rangle$ in a deterministic fashion? (i.e., w/out any probabilistic uncertainty).

Possible Answer #1: With a S-G device? NO!! (why not?)

Answer #2: Apply a Hamiltonian, $\hat{H}$, to the particle so that its wave function is rotated on Bloch Sphere from $|\psi\rangle$ to $|\psi'\rangle$ via Unitary Transf. $e^{-i\hat{H}t/\hbar}$! $|\psi'\rangle = e^{-i\hat{H}t/\hbar}|\psi\rangle$ (YES!

But how do we actually accomplish this in the Laboratory??

Answer: By turning on a magnetic Field ($\vec{B}$)!!

To understand this ⇒ Must understand behavior of electron spin in a $\vec{B}$-field (treatment for protons is identical!

What happens to electron spin when B-field is turned on?

⇒ Must solve Schr. Equation!!

The usual recipe for this (and for all Q.M. Problems!) is:

1. Solve time independent Schr. eq'n ($\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle$) to get allowed eigen-energies and corresponding eigen-states.
2. Use these solutions to help solve time-dependent Schr. eq'n: $\hat{H}|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle \longrightarrow |\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle$

So, where do we start? Must Find Hamiltonian!

For electron in $\vec{B}$-field, classical energy is $E = -\vec{\mu}\cdot\vec{B}$

⇒ Standard trick is to turn this into operator $\hat{H} = -\hat{\vec{\mu}}\cdot\vec{B}$

$\vec{B}$ is an applied Field, not an operator, $\vec{\mu}$ is an observable ⇒ turn into operator

$$\vec{\mu}_{electron} = \frac{-e}{m}\hat{\vec{S}} \Longrightarrow \boxed{\hat{H} = \frac{+e}{m}\hat{\vec{S}}\cdot\vec{B}}$$

Now must find eig-states, eig. energies of this $\hat{H}$.

Start by picking an orientation for $\vec{B}$. Usual choice: $\vec{B}\parallel\hat{z}$