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Quantum derivatives and the Schrödinger equation

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Abstract

We define a scale derivative for non-differentiable functions. It is constructed via quantum derivatives which take into account non-differentiability and the existence of a minimal resolution for mean representation. This justify heuristic computations made by Nottale in scale-relativity. In particular, the Schrödinger equation is derived via the scale-relativity principle and Newton's fundamental equation of dynamics.

1. Introduction

The aim of this article is to give a complete proof that the Schrödinger equation can be obtained from Newton's fundamental equation of dynamics in the scale-relativity setting developed by Nottale [1,11,14]. The articles [1,11,14] contain only a sketch of proof and are based on informal arguments.

The main "ingredient" of Nottale's work is a new "derivative", [1] that he calls the *scale derivative*, which applies to non-differentiable functions. Despite its importance, there exists no rigorous definition of this operator. In this article, we give a precise definition of the scale derivative, as well as its geometric interpretation. As a consequence, we can justify completely the computations made by Nottale in the articles [1,11,14].

The main problem is to define an extension of the classical differential calculus which has a clear physical meaning.

The starting point of our work is the following informal idea: Let γ be a given curve. From the physical view-point, we do not have access to γ, but to a "representation" of it, denoted γ_τ, which is always differentiable (up to a finite number of points) at a given scale of observation τ, and such that γ_τ converge to γ in C^0 topology. Of course, γ_τ is not always sufficient in order to describe the underlying physical process. In particular, if γ is non-differentiable, the fluctuations of γ_τ when τ goes to zero become non-negligible [2] contrary to what happens in the differentiable case.

This transition from a differentiable behaviour to a non-differentiable one when we follow γ_τ must be quantified. In this article, we introduce several concepts in the special case of graphs of functions $f(t)$ and for a representation given by the τ-mean function $f_\tau(t) = (1/2\tau)\int_{t-\tau}^{t+\tau} f(s)\,ds$.

In Section 2 we define the notion of τ-differentiability, which leads to a natural transition quantity for the differentiable–non-differentiable behaviour of $f_\tau(t)$ with respect to $f(t)$, called *minimal resolution* and denoted $\tau(f)$.

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[1] We will see that this terminology is not appropriate.

[2] Following Greene [8, p. 149] this is the basic reason for which differentiable (Riemanian) manifolds of Einstein's relativity theory cannot be used to describe the structure of space–time in quantum mechanics.

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Heuristically, we can say that for $\tau > \tau(f)$, the approximation of $f(t)$ by $f_\tau(t)$ is sufficient to describe the behaviour of f up to τ small perturbations, which is not the case for $\tau < \tau(f)$.

When $\tau < \tau(f)$, we must take into account the non-differentiability of the limiting function f. We then define left and right quantum derivatives, denoted $\Box_+ f/\Box t$ and $\Box_- f/\Box t$ respectively, which are nothing else than the derivatives of the right and left $\tau(f)$-mean function of f, i.e. $f_\sigma(t) = (\sigma/\tau(f)) \int_t^{t+\sigma\tau(f)} f(s)\,\mathrm{d}s$, $\sigma = \pm$. The fact to consider separately $f_+(t)$ and $f_-(t)$ is due to the non-differentiability of f.

The scale derivative, defined in Section 3 and denoted by $\Box f/\Box t$, is a complex operator which takes into account the two quantity $\Box_\sigma f/\Box t$, $\sigma = \pm$, in such a way that when f is differentiable $\Box f/\Box t = f'(t)$. This "gluing" property of $\Box/\Box t$ to $\mathrm{d}/\mathrm{d}t$ on the set of differentiable functions is a necessary constraint to be satisfied by all extended "differential" calculus. Paragraph 4 gives some properties of the scale derivative.

In Section 5, we generalize, following Nottale's scale relativity principle, Newton's fundamental equation of dynamics, by replacing the classical derivative by our scale derivative. We prove that the new equation leads to a generalized Schrödinger equation. This justify heuristic computations made by Nottale [1,11,14].

2. Non-differentiable functions and minimal resolution

In the following, f is a continuous function, defined on an open set I of $\mathbb{R}$. For all $\tau \in \mathbb{R}^{+*}$, we denote by f_τ the τ-mean function defined by

$$f_\tau(t) = (1/2\tau)\int_{t-\tau}^{t+\tau} f(s)\,\mathrm{d}s. \tag{1}$$

We call τ-oscillation of f the quantity

$$\mathrm{osc}_\tau f(t) = \sup\{f(t') - f(t''),\ t', t'' \in [t-\tau, t+\tau]\}. \tag{2}$$

We say that the graph of f is *fractal* according to Tricot [12] if the quantity

$$\frac{\mathrm{osc}_\tau f(t)}{\tau} \to +\infty \quad \text{when } \tau \to 0, \tag{3}$$

uniformly with respect to t, contrary to the differentiable case.

For $0 < \alpha \leqslant 1$, we denote

$$|f(t)|_{\alpha,\tau} = \sup_{s,s' \in [t-\tau, t+\tau],\, s \neq s'} \frac{|f(s) - f(s')|}{|s - s'|^\alpha}. \tag{4}$$

Remark 1. We have

$$\underline{|f|}_\alpha \leqslant |f(t)|_{\alpha,\tau} \leqslant |f|_\alpha, \tag{5}$$

where

$$\underline{|f|}_\alpha = \inf_{t \neq t' \in \bar{I}} \frac{|f(t) - f(t')|}{|t - t'|}, \quad |f|_\alpha = \sup_{t \neq t' \in \bar{I}} \frac{|f(t) - f(t')|}{|t - t'|}, \tag{6}$$

and $\bar{I}$ is the closure of I.

For all $t \in I$, we define $\alpha(t)$ such that

$$\alpha(t; f, \tau) = \sup\{\alpha > 0 \,|\, |f(t)|_{\alpha,\tau} \neq 0\}. \tag{7}$$

In the following, we do not write the explicit dependence of $\alpha(t; f, \tau)$ with respect to f and τ.

We denote

$$a_\tau f(t) = \frac{\mathrm{osc}_\tau f(t)}{2\tau |f(t)|_{\alpha(t),\tau}}. \tag{8}$$

We have

$$|f_\tau(t) - f(t)| \leqslant 2\tau |f(t)|_{\alpha(t),\tau} a_\tau f(t), \tag{9}$$

so that $a_\tau f(t)$ is a measure of the approximation of f by the differentiable function f_τ.

Remark 2. For a differentiable function, we have $\alpha(t) = 1$ for all $t \in I$ and $a_\tau f(t) \leqslant 1$ for all t and all τ.

We then have the following notion of τ-differentiability at a point t:

Definition 1. Let f be a continuous function, defined on a compact set I of $\mathbb{R}$, such that for all $t \in I$,

$$1 \geqslant \alpha(t) > 0. \tag{10}$$

Let $\tau > 0$ be given. We say that f is τ-differentiable at point $t \in I$ if

$$a_\tau f(t) \leqslant 1. \tag{11}$$

A large class of continuous functions satisfy condition (10). At least, Hölderian functions of order $0 < \alpha < 1$. Moreover, there exists explicit construction of continuous functions with prescribed Hölder regularity, as discussed in [16]. The simplest case is that of continuous function such that $\alpha(t) = \alpha$ for all $t \in I$. As an example, we can consider the Weierstrass function which possesses a uniform Hölder regularity [12].

Remark 3. The concept of τ-differentiability can be understood as a way to characterize when the fluctuations of f are "small" with respect to the mean function f_τ. Of course, the notion of "smallness" depends on a normalization, which is fixed. In our case, this normalization is contained in the choice of the number 1 for the upperbound of $a_{\tau,\alpha} f$ in (11), by comparison with the differentiable case. Different kinds of normalization can be taken, leading to different notions of non-differentiability. This is just a matter of choice.

We denote by $\tau(f)(t)$ the minimal order of τ-differentiability at point t:

$$\tau(f)(t) = \inf\{\tau \geqslant 0 \,|\, f \text{ is } \tau\text{-differentiable at point } t\}. \tag{12}$$

Definition 2. Let f be a continuous real valued function defined on an open interval $I \subset \mathbb{R}$. We call minimal resolution the quantity

$$\tau(f) = \inf_{t \in \overline{I}} \tau(f)(t). \tag{13}$$

We remark that for all $\lambda \in \mathbb{R}$, we have

$$\tau(\lambda f) = \tau(f) \quad \text{and} \quad \tau(f + \lambda) = \tau(f). \tag{14}$$

If f is a fractal function (see (3)), then $\tau(f) > 0$. In fact, we have a more general result:

Lemma 1. *If a continuous function f is differentiable then $\tau(f) = 0$.*

Proof. This follows from Remark 2. □

As a consequence, if $\tau(f) > 0$, then f is an everywhere non-differentiable function. Of course, if f is differentiable on a given subset J of I, then $\tau(f) = 0$, despite the fact that $\tau(f)(t) > 0$ for all $t \in I \setminus J$. In the following, we consider only continuous everywhere non-differentiable functions.

3. Quantum derivatives and the scale derivative

We define left and right *quantum derivatives*.

Definition 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that its minimal resolution $\tau(f)$ satisfies $\tau(f) > 0$. We call right and left quantum derivative the quantities

$$\frac{\square_+ f}{\square t}(t) = \frac{f(t+\tau) - f(t)}{\tau},$$
$$\frac{\square_- f}{\square t}(t) = \frac{f(t) - f(t-\tau)}{\tau}, \tag{15}$$

respectively (where $\tau = \tau(f)$). If $\tau = 0$, then $\frac{\square_\sigma f}{\square t}(t) = \lim_{\tau\to 0} \frac{f(t+\tau)-f(t)}{\tau}$.

For a differentiable function, we have $\frac{\square_+ f}{\square t}(t) = \frac{\square_- f}{\square t}(t) = f'(t)$.

Remark 4. We refer to [17] for a careful study of such kind of difference operators in the context of what they call time scales calculus.

We define *quantum functions* $f_+(t)$ and $f_-(t)$ as

$$f_+(t) = \frac{1}{\tau}\int_t^{t+\tau} f(s)\,\mathrm{d}s,$$
$$f_-(t) = \frac{1}{\tau}\int_{t-\tau}^{t} f(s)\,\mathrm{d}s, \tag{16}$$

respectively.

We have

$$\frac{\square_\sigma f}{\square t}(t) = f'_\sigma(t), \quad \sigma = \pm. \tag{17}$$

Remark 5. As $\frac{\square_+ f}{\square t}(t)$ and $\frac{\square_- f}{\square t}(t)$ are continuous, the functions $f_+(t)$ and $f_-(t)$ are well defined.

Using the quantum functions $f_+(t)$ and $f_-(t)$, we can write f as

$$f(t) = f_+(t) + \xi_+^f(t), \quad f(t) = f_-(t) + \xi_-^f(t), \tag{18}$$

where ξ_+^f and ξ_-^f are non-differentiable functions representing fluctuations with respect to the right (resp. left) mean function. In the following, we will denote ξ_+^f and ξ_-^f by ξ_+ and ξ_- if no confusion is possible.

Lemma 2. *Let f be a continuous function such that $\tau(f) > 0$. Then for h sufficiently small, we have*

$$f(t+\sigma h) = f(t) + \sigma\frac{\square_\sigma f}{\square t}(t)h + (\xi_\sigma(t+h) - \xi_\sigma(t)) + \mathrm{o}(h), \tag{19}$$

where $\frac{\square_\sigma f}{\square t}(t) = \frac{f(t+\sigma\tau(f))-f(t)}{\sigma\tau(f)}$, $\sigma = \pm$.

Proof. We have $f(t+\sigma h) = f_\sigma(t+\sigma h) + \xi_\sigma(t+\sigma h) + \mathrm{o}(h)$ for $\sigma = \pm$. A first order Taylor's expansion of $f_\sigma(t+\sigma h)$ gives

$$f_\sigma(t+\sigma h) = f_\sigma(t) + \sigma f'_\sigma(t)h + \mathrm{o}(h). \tag{20}$$

Using (17), we obtain for h sufficiently small,

$$f(t+h) = f_+(t) + \frac{\square_+ f}{\square t}(t)h + \xi_+(t+h) + \mathrm{o}(h),$$
$$f(t-h) = f_-(t) + \frac{\square_- f}{\square t}(t)h + \xi_-(t-h) + \mathrm{o}(h).$$

Replacing $f_\sigma(t)$ by $f(t) - \xi_\sigma(t)$, we obtain (19). □

In the following, we denote

$$r_\sigma^f(t,h) = \xi_\sigma^f(t+\sigma h) - \xi_\sigma^f(t) \tag{21}$$

for $\sigma = \pm$, or $r_\sigma(t,h)$ if no confusion is possible.

Definition 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $\tau(f)$ its minimal resolution. We call scale derivative of f at point t the quantity

$$\frac{\Box f}{\Box t}(t) = \frac{1}{2}\left(\frac{\Box_+ f}{\Box t}(t) + \frac{\Box_- f}{\Box t}(t)\right) - \mathrm{i}\frac{1}{2}\left(\frac{\Box_+ f}{\Box t}(t) + \frac{\Box_- f}{\Box t}(t)\right), \quad \mathrm{i}^2 = -1. \tag{22}$$

Remark 6. We use the terminology of scale derivative for the operator $\frac{\Box}{\Box t}$ only to be coherent with Nottale's terminology [1,11]. However, as quantum derivatives, the scale derivative is not a derivation on the set of continuous functions, i.e. it does not satisfy the Liebniz rule. [3]

When f is differentiable, we obtain the classical derivative. The real part of the scale derivative is the derivative of the τ-mean function of f, f_τ. We have

$$\frac{f_+(t) + f_-(t)}{2} = f_\tau(t).$$

In our computations about the Schrödinger equation, we will need the following definition of the scale derivative for complex valued functions:

Let $f : \mathbb{R} \to \mathbb{C}$, $t \mapsto f(t)$, be a continuous function. We have $f(t) = \mathrm{Re}(f(t)) + \mathrm{i}\,\mathrm{Im}(f(t))$ where Re and Im are the real and imaginary part of $f(t)$ respectively. The functions $\mathrm{Re}(f(t))$ and $\mathrm{Im}(f(t))$ are continuous real valued functions. We define the scale derivative of f as

$$\frac{\Box f}{\Box t}(t) = \frac{\Box \mathrm{Re}(f)}{\Box t}(t) + \mathrm{i}\frac{\Box \mathrm{Im}(f)(t)}{\Box t}. \tag{23}$$

4. Consequences of non-differentiability

We keep the notations of Section 3.

4.1. Main lemma

In order to derive the Schrödinger equation from Newton's fundamental equation of dynamics in Section 5, we need to compute the scale derivative of a composed function of the form $f(x(t), t)$ where $f(x, t)$ is a differentiable function, and $x(t)$ is not. The following lemma gives the formula.

Lemma 3. *Let $f(x, t)$ be a C^3 function. Let $x(t)$ be a continuous function such that $\tau(x) > 0$ and*

$$x(t + \sigma h) = x(t) + \sigma \frac{\Box_\sigma x}{\Box t}(t)h + r^x_\sigma(t) \tag{24}$$

for $h > 0$ sufficiently small, $\sigma = \pm$.

We consider the function $g(t) = f(x(t), t)$. We have

(i) $\tau(g) > 0$.

For all h sufficiently small, we denote

$$g(t + \sigma h) = g(t) + \sigma \frac{\Box_\sigma g}{\Box t}(t)h + r^g_\sigma(t, h), \quad \sigma = \pm. \tag{25}$$

We assume that for $\sigma = \pm$, we have

(*) *the function $(r^x_\sigma(t, h))^2$ admits a right derivative at point $h = 0$ for all t,*
(**) *we have $\left[r^g_\sigma(t, h) - \frac{\partial f}{\partial x} r^x_\sigma(t, h)\right]/h \to 0$ when $h \to 0^\sigma$.*

[3] An operator D on an algebra A is a derivation if $\forall x, y \in A$ we have $D(x \cdot y) = \mathrm{D}x \cdot y + x \cdot \mathrm{D}y$ which is usually called Liebniz identity (see [10]).

Then, we have

$$\text{(ii)}\quad \frac{\Box_\sigma g}{\Box t} = \frac{\partial f}{\partial x}(x(t),t)\frac{\Box_\sigma x}{\Box t}(t) + \frac{\partial f}{\partial t}(x(t),t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x(t),t)a_\sigma(t), \tag{26}$$

where $a_+(t)$ *(resp.* $a_-(t)$*) is the right derivative at point* $h = 0$ *of* $r_+^2(t,h)$ *(resp.* $r_-^2(t,h)$*).*

Remark 7. If f is a flat function at point $(x(t),t)$ then $(\partial f/\partial x)(x(t),t) = 0$ and assumption (**) is equivalent to the differentiability of $r_\sigma(t,h)$, $\sigma = \pm$ at $h = 0$, which is not possible as $\tau(x) > 0$.

Proof. For (i) this follows easily from Lemma 1. For (ii), we have

$$g(t+h) = f(x(t+h), t+h) = f\left(x(t) + \frac{\Box_+ x}{\Box t}(t)h + o(h) + r_+^x(t,h), t+h\right), \tag{27}$$

using (19).

As $r_+^x(t,h) \to 0$ when $h \to 0$, we have, doing a Taylor's expansion of f in the neighborhood of $(x(t),t)$,

$$\begin{aligned} g(t+h) = {} & g(t) + \frac{\partial f}{\partial x}(x(t),t)\left(\frac{\Box_+ x}{\Box t}(t)h + o(h) + r_+^x(t,h)\right) + \frac{\partial f}{\partial t}(x(t),t)h \\ & + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(x(t),t)\left(\frac{\Box_+ x}{\Box t}(t)h + o(h) + r_+^x(t,h)\right)^2\right. \\ & \left. + 2\frac{\partial^2 f}{\partial x \partial t}(x(t),t)\left(\frac{\Box_+ x}{\Box t}(t)h + o(h) + r_+^x(t,h)\right)h + \frac{\partial^2 f}{\partial t^2}(x(t),t)h^2\right) + \cdots \end{aligned} \tag{28}$$

As $(r_+^x)^2(t,h)$ is differentiable at point h, we have $(r_+^x)^2(t,h) = a_+(t)h + o(h)$. By factorizing terms of order 1 in h, we obtain

$$g(t+h) = g(t) + \left(\frac{\partial f}{\partial x}(x(t),t)\frac{\Box_+ x}{\Box t}(t) + \frac{\partial f}{\partial t}(x(t),t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x(t),t)a_+(t)\right)h + o(h) + R_g^+(t,h), \tag{29}$$

where

$$R_g^+(t,h) = \frac{\partial f}{\partial x}(x(t),t)r_+^x(t,h) \tag{30}$$

is non-differentiable and $r_+^g(t,0) = 0$.

We deduce

$$\begin{aligned} & \frac{\Box_+}{\Box t} f(x(t),t) - \left(\frac{\partial f}{\partial x}(x(t),t)\frac{\Box_+ x}{\Box t}(t) + \frac{\partial f}{\partial t}(x(t),t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x(t),t)a_+(t)\right) \\ & \quad = \frac{1}{h}\left(r_g^+(t,h) + o(h) - R_g^+(t,h)\right) \quad \forall h > 0. \end{aligned} \tag{31}$$

By replacing $R_g(t,h)$ by (30) and taking the limit in (31), we obtain, using (**),

$$\frac{\Box_+}{\Box t} f(x(t),t) = \frac{\partial f}{\partial x}(x(t),t)\frac{\Box_+ x}{\Box t}(t) + \frac{\partial f}{\partial t}(x(t),t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x(t),t)a_+(t), \tag{32}$$

where $a_+(t)$ is the derivative of $(r_+^x)^2(t,h)$ at point $h = 0$.

Similar computations allow us to prove that

$$\frac{\Box_-}{\Box t} f(x(t),t) = \frac{\partial f}{\partial x}(x(t),t)\frac{\Box_- x}{\Box t}(t) + \frac{\partial f}{\partial t}(x(t),t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x(t),t)a_-(t), \tag{33}$$

where $a_-(t)$ is the derivative of $(r_-^x)^2(t,h)$ at point $h = 0$. □

About Ito's stochastic calculus. As a consequence, the non-differentiability of $x(t)$ introduces additional spatial terms in the derivative of $f(x(t),t)$ with respect to the differentiable case. This formula is similar to Ito's formula in stochastic calculus [15]. However, Ito's formula is obtained under probabilistic assumptions. In Lemma 4, this follows from the geometric assumption that f is non-differentiable. For more details about our scale derivative and Ito's formula, we refer to [2,3].

4.2. The complex case

Let

$$C: \begin{array}{ccc} \mathbb{R}\times\mathbb{R} & \to & \mathbb{C} \\ (x,t) & \mapsto & C(x,t) \end{array}.$$

We denote $A(x,t) = \mathrm{Re}C(x,t)$ and $B(x,t) = \mathrm{Im}C(x,t)$, then $C(x,t) = A(x,t) + \mathrm{i}B(x,t)$.

Let

$$x: \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ t & \mapsto & x(t) \end{array}$$

be a continuous function such that its minimal resolution $\tau(f)$ satisfies $\tau(f) > 0$. We define the functions $x_+(t)$, $x_-(t)$, $\xi_+(t)$ and $\xi_-(t)$ as in Section 3.

We denote by $a_\sigma(t)$ the right derivative of $(\xi_\sigma(t+\sigma h) - \xi_\sigma(t))^2$ at point $h = 0$.

Lemma 4. *The scale derivative of the function*

$$\mathscr{C}: \begin{array}{ccc} \mathbb{R} & \to & \mathbb{C} \\ t & \mapsto & C(x(t),t) \end{array}$$

is

$$\frac{\Box\mathscr{C}}{\Box t} = \frac{\partial C}{\partial t} + \frac{\Box x}{\Box t}\frac{\partial C}{\partial x} + \frac{1}{2}a(t)\frac{\partial^2 C}{\partial x^2}, \tag{34}$$

where

$$a(t) = \left(\frac{a_+(t) - a_-(t)}{2}\right) - \mathrm{i}\left(\frac{a_+(t) + a_-(t)}{2}\right). \tag{35}$$

Proof. We denote by $\mathscr{A}(t)$ and $\mathscr{B}(t)$ the functions $A(x(t),t)$ and $B(x(t),t)$ respectively. By Lemma 4, we have

$$\frac{\Box_\sigma\mathscr{A}}{\Box t} = \frac{\partial A}{\partial x}\frac{\Box_\sigma x}{\Box t} + \frac{\partial A}{\partial t} + \sigma\frac{1}{2}a_\sigma(t)\frac{\partial^2 A}{\partial x^2}.$$

We deduce

$$\frac{\Box\mathscr{A}}{\Box t} = \frac{\partial A}{\partial x}\frac{1}{2}\left(\frac{\Box_+ x}{\Box t} + \frac{\Box_- x}{\Box t}\right) + \frac{\partial A}{\partial t} + \frac{1}{2}\frac{\partial^2 A}{\partial x^2}\frac{1}{2}(a_+(t) - a_-(t)) - \mathrm{i}\left(\frac{\partial A}{\partial x}\frac{1}{2}\left(\frac{\Box_+ x}{\Box t} - \frac{\Box_- x}{\Box t}\right) + \frac{1}{2}\frac{\partial^2 A}{\partial x^2}\frac{1}{2}(a_+(t) + a_-(t))\right),$$

that is

$$\frac{\Box\mathscr{A}}{\Box t} = \frac{\partial A}{\partial x}\frac{\Box x}{\Box t} + \frac{\partial A}{\partial t} + \frac{1}{2}a(t)\frac{\partial^2 A}{\partial x^2}$$

with $a(t) = \left(\frac{a_+(t)-a_-(t)}{2}\right) - \mathrm{i}\left(\frac{a_+(t)+a_-(t)}{2}\right)$.

We obtain a similar formula for $\mathscr{B}(t)$. Hence, by definition, we have

$$\frac{\Box\mathscr{C}}{\Box t} = \frac{\Box x}{\Box t}\left(\frac{\partial A}{\partial x} + \mathrm{i}\frac{\partial B}{\partial x}\right) + \left(\frac{\partial A}{\partial t} + \mathrm{i}\frac{\partial B}{\partial t}\right) + \frac{1}{2}a(t)\left(\frac{\partial^2 A}{\partial x^2} + \mathrm{i}\frac{\partial^2 B}{\partial x^2}\right).$$

We then obtain

$$\frac{\Box\mathscr{C}}{\Box t} = \frac{\Box x}{\Box t}\frac{\partial C}{\partial x} + \frac{1}{2}a(t)\frac{\partial^2 C}{\partial x^2} + \frac{\partial C}{\partial t},$$

which concludes the proof. □

4.3. About the regularity assumption (*)

Feynman and Hibbs [7] have proved that generic trajectories of quantum particles are continuous non-differentiable curves. However, there exists a quadratic velocity, that is, the quantity

$$\lim_{x\to x'} \frac{(f(x)-f(x'))^2}{x-x'} \text{ exists.} \tag{36}$$

Hence, the following quantities

$$\frac{(f(t+h)-f(t))^2}{h} \quad \text{and} \quad \frac{(f(t)-f(t-h))^2}{h} \tag{37}$$

keep sense when $h > 0^+$, and are equals.

For all $h > 0$, we have

$$\frac{r_+^2(t,h)}{h} = \frac{\left(f(t+h)-f(t)-\left(\frac{\Box_+ f}{\Box t}h+o(h)\right)\right)^2}{h}. \tag{38}$$

When $h \to 0^+$, we obtain

$$\lim_{h\to 0^+} \frac{r_+^2(t,h)}{h} = \lim_{h\to 0^+} \frac{(f(t+h)-f(t))^2}{h}. \tag{39}$$

Similar computations prove that

$$\lim_{h\to 0^-} \frac{r_-^2(t,h)}{h} = \lim_{h\to 0^-} \frac{(f(t)-f(t-h))^2}{h}.$$

We denote by $a_+(t)$ (resp. $a_-(t)$) the right derivative of $r_+^2(t,h)$ (resp. $r_-^2(t,h)$). As the quadratic velocity is well defined, we must have

$$a_+(t) = a_-(t). \tag{40}$$

Assumption (*) is then satisfied by functions describing quantum trajectories.

Remark 8

- For the Browninan motion, Einstein [9] has proved that $f(t+h)-f(t) \approx h^{1/2}$ for $h > 0$, which is in accordance with (*).
- The existence of a quadratic velocity is equivalent to 1/2-right differentiability of f following [4].

5. Scale relativity principle and Schrödinger equation

In [1,11,14], Nottale announce that the Schrödinger equation can be obtained from the classical Newton's equation of dynamics using a quantization procedure which comes from the *scale relativity theory*. The scale relativity theory is developed by Nottale since 1980. Its aim is to generalize Einstein's relativity principle in order to derive quantum mechanics from a first principle. We refer to his work for more details [1]. The quantization procedure is based on a *generalized Euler–Lagrange equation* coming from Nottale's theory and the use of the scale derivative instead of the classical derivative. The computations made by Nottale in [1,11,14] are informal and based on heuristic arguments. Using the scale derivative defined in the previous paragraph, we give a complete and detailed proof of his approach.

5.1. Action functional and wave function

Let $x : \mathbb{R} \to \mathbb{R}$, $t \mapsto x(t)$, be a continuous, non-differentiable function, describing the trajectory of a quantum particle of mass m. Let

$$\begin{array}{cccc} v: & \mathbb{R} & \to & \mathbb{C} \\ & t & \mapsto & v(t) = \dfrac{\Box x}{\Box t} \end{array}$$

be its velocity. Let

$$\begin{array}{llll}\Phi: & \mathbb{R}\times\mathbb{R} & \to & \mathbb{C} \\ & (x,t) & \mapsto & \Phi(x,t)\end{array} \tag{41}$$

be a differentiable function, called *scalar potential*.

The action functional is then defined by

$$\begin{array}{llll}\mathscr{L}: & \mathbb{R}\times\mathbb{C}\times\mathbb{R} & \to & \mathbb{C} \\ & (x,v,t) & \mapsto & \frac{1}{2}mv^2-\Phi(x,t)\end{array}. \tag{42}$$

We note that the map $\mathscr{L}(x,v,t)$ is differentiable with respect to x and v.

Scale assumption. We assume, following Nottale [1], that equation of motion for particles is given by the following *Euler–Lagrange generalized equation*:

$$\frac{\Box}{\Box t}\left(\frac{\partial\mathscr{L}}{\partial v}\right)=\frac{\partial\mathscr{L}}{\partial x}. \tag{43}$$

Nottale deduce this equation informally via his *scale relativity principle*. We refer to his work [1,5,6] for more details.

We then have

$$m\frac{\Box v}{\Box t}=-\frac{\partial\Phi}{\partial x}. \tag{44}$$

We call this equation, *fundamental equation of dynamics*, by analogy with Newton's classical equation.

The *momentum* is defined by $p=\frac{\partial\mathscr{L}}{\partial v}$, which gives $p=mv$. We introduce an *action* A as

$$\begin{array}{llll}A: & \mathbb{R}\times\mathbb{R} & \to & \mathbb{C} \\ & (x,t) & \mapsto & A(x,t)\end{array}, \tag{45}$$

which is a differentiable function, related to the momentum via the relation $p=\frac{\partial A(x,t)}{\partial x}$. We then obtain $v=\frac{1}{m}\frac{\partial A}{\partial x}$.

We can introduce a function

$$\begin{array}{llll}\psi: & \mathbb{R}\times\mathbb{R} & \to & \mathbb{C} \\ & (x,t) & \mapsto & \psi(x,t)\end{array}, \tag{46}$$

differentiable, such that

$$\psi(x,t)=\mathrm{e}^{\frac{\mathrm{i}A(x,t)}{2m\gamma}}, \tag{47}$$

where $\gamma\in\mathbb{R}$ is a normalization constant to be determined.

This function is of course the *wave function* of a particle. We note that $A(x,t)=-2m\gamma\mathrm{i}\ln\psi(x,t)$ and $v=v(x,t)=-\mathrm{i}2\gamma\frac{\partial\ln\psi}{\partial x}$, where ln is the complex logarithm.

Remark 9. We obtain the classical correspondence principle of quantum mechanics for momentum and energy, that is

$$p=-2\mathrm{i}m\gamma\frac{\partial\psi}{\partial x}\frac{1}{\psi},\quad E=2\mathrm{i}m\gamma\frac{\partial\psi}{\partial t}\frac{1}{\psi}. \tag{48}$$

5.2. Schrödinger equation

Using the wave function, the fundamental equation of dynamics looks like

$$2\mathrm{i}\gamma m\frac{\Box}{\Box t}\left(\frac{\partial}{\partial x}(\ln\psi)\right)=\frac{\partial\Phi}{\partial x}. \tag{49}$$

Lemma 5. *The fundamental equation of dynamics is equivalent to*

$$-\mathrm{i}2\gamma m\left(\mathrm{i}\gamma+\frac{a(t)}{2}\right)\left(\frac{\partial\psi}{\partial x}\right)^2\frac{1}{\psi^2}+\mathrm{i}2\gamma m\frac{\partial\ln\psi}{\partial t}+\mathrm{i}\gamma a(t)\frac{\partial^2\psi}{\partial x^2}\frac{1}{\psi}=\Phi+\alpha(x). \tag{50}$$

Proof. The fundamental equation of dynamics is

$$2\mathrm{i}\gamma m \frac{\Box}{\Box t}\left(\frac{\partial \ln\psi}{\partial x}\right) = \frac{\partial \Phi}{\partial x}.$$

We denote $\Xi(t) = \xi(x(t), t) = \frac{\partial \ln\psi}{\partial x}(x(t), t)$. We have, using Lemma 5,

$$\frac{\Box}{\Box t}\left(\frac{\partial}{\partial x}(\ln\psi)\right) = \frac{\Box \Xi}{\Box t} = \frac{\partial \xi}{\partial x}\frac{\Box x}{\Box t} + \frac{\partial \xi}{\partial t} + \frac{1}{2}a(t)\frac{\partial^2 \xi}{\partial x^2}.$$

A simple computation gives $\frac{\partial \xi}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial \ln\psi}{\partial t}\right)$ and

$$\frac{\partial \xi}{\partial x}\frac{\Box x}{\Box t} = -2\mathrm{i}\gamma \frac{\partial \xi}{\partial x}\xi,$$

by definition of $v = \frac{\Box x}{\Box t}$ as function of ψ. We then have

$$\frac{\partial \xi}{\partial x}\frac{\Box x}{\Box t} = -\mathrm{i}\gamma \frac{\partial \xi^2}{\partial x} = -\mathrm{i}\gamma \frac{\partial}{\partial x}\left(\frac{1}{\psi^2}\left(\frac{\partial \psi}{\partial x}\right)^2\right).$$

Moreover, we have

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial^2 \psi}{\partial x^2}\frac{1}{\psi} - \left(\frac{\partial \psi}{\partial x}\right)^2\frac{1}{\psi^2}\right).$$

We deduce, by gathering these terms

$$\frac{\Box \Xi}{\Box t} = \frac{\partial}{\partial x}\left(-\left(\mathrm{i}\gamma + \frac{a(t)}{2}\right)\left(\frac{\partial \psi}{\partial x}\right)^2\frac{1}{\psi^2} + \frac{a(t)}{2}\frac{\partial^2 \psi}{\partial x^2}\frac{1}{\psi} + \frac{\partial \ln\psi}{\partial t}\right).$$

By replacing in the fundamental equation of dynamics, we obtain

$$\frac{\partial}{\partial x}\left(-2\mathrm{i}\gamma m\left(\mathrm{i}\gamma + \frac{a(t)}{2}\right)\left(\frac{\partial \psi}{\partial x}\right)^2\frac{1}{\psi^2} + \mathrm{i}\gamma m a(t)\frac{\partial^2 \psi}{\partial x^2}\frac{1}{\psi} + 2\mathrm{i}\gamma m\frac{\partial \ln\psi}{\partial t}\right) = \frac{\partial \Phi}{\partial x}.$$

We conclude the proof by integration. □

As a particular case, when the non-differentiability of $x(t)$ is uniform, we obtain the classical Schrödinger equation.

Corollary 1. *Let $x(t)$ be a continuous, non-differentiable function such that*

$$a(t) = -\mathrm{i}2\gamma. \tag{51}$$

Then the fundamental equation of dynamics takes the form

$$\mathrm{i}2\gamma m\frac{\partial \psi}{\partial t} + 2\gamma^2 m\frac{\partial^2 \psi}{\partial x^2} = (\Phi + \alpha(x))\psi. \tag{52}$$

We can always choose a solution of (52) *such that $\alpha(x) = 0$. In this case, when*

$$\gamma = \frac{\bar{h}}{2m}, \tag{53}$$

where $\bar{h}$ is the Planck constant, we obtain the classical Schrödinger equation

$$\mathrm{i}\bar{h}\frac{\partial \psi}{\partial t} + \frac{\bar{h}^2}{2m}\frac{\partial^2 \psi}{\partial x^2} = \Phi\psi. \tag{54}$$

Proof. The choice of $a(t)$ allows us to cancel the term $\left(\frac{\partial \psi}{\partial x}\right)^2\frac{1}{\psi^2}$. In this case, by replacing $a(t)$ and remarking that $\frac{\partial \ln\psi}{\partial t} = \frac{\partial \psi}{\partial t}\frac{1}{\psi}$, we obtain Eq. (52).

Let ψ be a solution of (52). We search for a function $\tilde{\psi}$ solution of the equation

$$i2\gamma m\frac{\partial\tilde{\psi}}{\partial t}+2\gamma^2 m\frac{\partial^2\tilde{\psi}}{\partial x^2}=\Phi\tilde{\psi} \tag{55}$$

of the form

$$\tilde{\psi}=e^{i\frac{A(x,t)}{2m\gamma}+\theta(x)}=\psi(x,t)\Theta(x), \tag{56}$$

where

$$\Theta(x)=e^{\theta(x)}.$$

That is, we modify the phase of the wave function ψ.

We then have

$$\begin{aligned}\frac{\partial\tilde{\psi}}{\partial x}&=\frac{\partial\psi}{\partial x}\Theta+\psi\Theta',\\ \frac{\partial^2\tilde{\psi}}{\partial x^2}&=\frac{\partial^2\psi}{\partial x^2}\Theta+2\frac{\partial\psi}{\partial x}\Theta'+\psi\Theta'',\\ \frac{\partial\psi}{\partial t}&=\frac{\partial\psi}{\partial t}\Theta,\end{aligned} \tag{57}$$

where $\Theta'(x)$ and $\Theta''(x)$ are the first and second derivative of $\Theta(x)$.

By replacing in (55), we obtain

$$i2\gamma m\frac{\partial\psi}{\partial t}\Theta+2\gamma^2 m\left(\frac{\partial^2\psi}{\partial x^2}\Theta+2\frac{\partial\psi}{\partial x}\Theta'+\psi\Theta''\right)=\Phi\psi\Theta. \tag{58}$$

We deduce an ordinary differential equation in Θ of the form

$$\Theta\left(i2\gamma m\frac{\partial\psi}{\partial t}+2\gamma^2 m\frac{\partial^2\psi}{\partial x^2}-\Phi\psi\right)+4\gamma^2 m\frac{\partial\psi}{\partial x}\Theta'+2\gamma^2 m\psi\Theta''=0. \tag{59}$$

As ψ is a solution of (52), we have

$$\Theta\alpha(x)\psi+4\gamma^2 m\frac{\partial\psi}{\partial x}\Theta'+2\gamma^2 m\psi\Theta''=0. \tag{60}$$

This differential equation has always a solution. Hence, we can always choose a solution of (52) such that $\alpha(x)=0$.

The choice of γ in order to obtain Eq. (54) is then done by identification. □

Remark 10. Our derivation of the Schrödinger equation is done under the scale assumption, which follows from Nottale's physical concept of scale relativity principle. We refer to [13, pp. 254–257] for a completely different proof.

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